Mathematics *for* Today

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MATHEMATICS FOR TODAY

By

MILTON EZRA LAZERTE
Professor of Educational Psychology and Director of the School of Education, University of Alberta
DEAN of FAC. of Ed. U of A.
and
WILLIAM BETZ
Author of Junior Mathematics for Today, etc.

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PREFACE

The number of pupils remaining in school nowadays through the lower secondary grades is a large percentage of the total school enrolment. If these pupils are to be interested in mathematics and if mathematics is to contribute to their education as it should, it must have a vital connection with their experience and it must be meaningful and interpretative. It must not be solely preparatory to more advanced mathematics; it must have a content which will be worth while for *all* students. *Mathematics for Today* will give valuable training in both algebra and geometry to the pupil who leaves school at the end of the course and at the same time will give the proper background for those continuing the study of mathematics.

The authors have not juxtaposed pages of algebra and geometry, but have so organized the content that while the student focuses attention upon these subjects one at a time, he is given repeated opportunity to see the relationships that exist between the two phases of mathematics.

Throughout the course the goal should be real understanding and mastery rather than a merely imitative performance. While plenty of drill is provided on the essentials, this text emphasizes the understanding of relationships rather than mere manipulation. Much of the algebra section is devoted to the relationships expressed in tables, formulas, equations, and graphs. The same stress on dynamic relationships will be noted throughout the part of the book dealing with geometry. Geometric figures are not isolated and static. They change gradually from one form to another. They are presented as parts of a continuum.

Problem-solving receives much emphasis. The problems are used to challenge thought and many problem-solving

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aids are given throughout the book. The authors have had in mind pupils who are to read and interpret rather than memorize and reproduce.

In the geometry the fundamental propositions on congruence, on parallels, and on relationships in the isosceles triangle are assumed. Pupils are not interested in so-called logical proofs of statements that appear obvious to them. Although the content of the early sections is treated informally, the rigor of treatment gradually increases. It is better for the pupil to make assumptions, if he knows he is making them, and then go on to the consideration of real problems which he will view in the light of his assumptions, than it is for him to begin with what he thinks are a few facts and try to reason at levels beyond his years to other unchallenged truths.

As an aid to the teacher of elementary mathematics the following list of reference books is suggested:

1. DURELL. The Teaching of Elementary Algebra. G. Bell and Sons, Ltd., London, 1927.
2. FORDER, H. G. A School Geometry. Cambridge University Press, 1930.
3. GODFREY and SIDDONS. The Teaching of Elementary Mathematics. Cambridge University Press, 1931.
4. The National Council of Teachers of Mathematics. The Fifth Yearbook (1930): The Teaching of Geometry. Bureau of Publications, Teachers College, Columbia University.
5. The National Council of Teachers of Mathematics. The Seventh Yearbook (1932): The Teaching of Algebra. Bureau of Publications, Teachers College, Columbia University.
6. NUNN, T. P. The Teaching of Algebra (including Trigonometry). Longmans, Green and Co., Ltd., 1927.
7. A Report Prepared for the Mathematical Association: The Teaching of Algebra in Schools. G. Bell and Sons, Ltd., London, 1934.
8. A Report Prepared for the Mathematical Association: The Teaching of Geometry in Schools. G. Bell and Sons, Ltd., London, 1934.
9. SMITH and REEVE. The Teaching of Junior High School Mathematics. Ginn and Company, 1927.

The authors feel that *Mathematics for Today*, with its departures from the old conventional content, will be welcomed by pupils and teachers alike.

M. E. L.
W. B.

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The whetstone of witte.

whiche is the seconde parte of Arithmetike: containyng thextraction of Rootes: The Cossike practise, with the rule of *Equation*: and the woorkes of *Surde* Nombers.

Though many stones doe beare greate price,
The whetstone *is for exercise*
As needefull, and in woorke as straunge
Dulle thinges and harde it will so chaunge,
And make them sharpe, to right good vse
All artesmen knowe, thei can not chuse,
But vse his helpe: yet as men see,
Noe sharpenesse semeth in it to bee.
The grounde of artes *did brede this stone*
His vse is greate, and more then one
Here if you list your wittes to whette,
Moche sharpenesse therby shall you gette
Dulle wittes hereby doe greately mende,
Sharpe wittes are fined to their fulle ende.
Now proue, and praise, as you doe finde,
And to your self be not vnkinde.

Title Page to *The Whetstone of Witte, which is the seconde parte of Arithmetike: containyng thextraction of Rootes: The Cossike practise with the rule of Equation: and the woorkes of Surde Nombers.* This is the first Algebra in the English language and the volume from which this photograph was made is now in the Bodleian Library at Oxford

MATHEMATICS FOR TODAY

PART ONE · *Algebra for Today*

CHAPTER I

HOW LETTERS ARE USED IN ALGEBRA

1. The Shorthand of Algebra. In beginning the study of algebra, you naturally wish to know as soon as possible "what it is all about." It may help you to keep in mind that algebra is a kind of *shorthand*, by means of which it is possible to solve many mathematical problems more easily and quickly.

Unlike *commercial shorthand*, algebra uses the letters of the alphabet and certain symbols for the purposes of *mathematical shorthand*. In your previous arithmetic work it is probable that you have already used abbreviations such as l for the length of a rectangle, w for its width, r for the radius of a circle, and so on. When you did that you really commenced the use of algebra.

2. Using the Shorthand of Algebra. We shall now try to learn how letters are used in algebra. Suppose that one of the windows of your schoolroom has two equal panes of glass and that each of these panes is h inches high. What is the total height of the window?

Solution. Evidently we can write:

Total height = h plus h. This may be shortened as follows: $t = h + h$,

or $t = 2 \times h$,

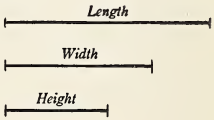
or $t = 2\,h$.

If h is measured in inches, then the result t will be in inches. If h is measured in feet, then t will be in feet also. Usually the unit is left out until the answer is written down, but you must be careful not to confuse the units with which you are working.

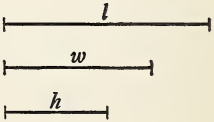
This result does not tell the numerical value of the total height, but simply that it is made up of two equal heights.

Exercises. *Using Letters*

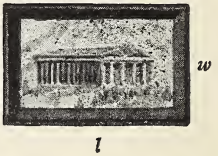
1. To find the size of your schoolroom, you would have to measure its *length*, its *width*, and its *height*. Draw three lines to represent these dimensions.



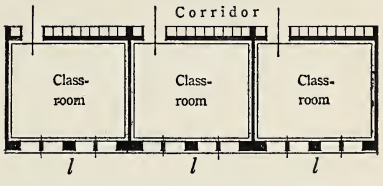
2. We may abbreviate the words *length*, *width*, and *height* by using the letters l, w, and h. In your diagram for Exercise 1, replace the words by letters.



3. Make a list of rectangular objects in the room, such as the floor, the blackboard, a window, a picture. Draw a rectangle representing one of these objects, such as the picture shown at the right. Denote its length and its width, in terms of the same unit, by l and w.



4. On one corridor of a school building there were three classrooms, all having the same length, as shown in the figure. If the length of one room is denoted by l, what is the combined length of the three rooms? (Refer to the illustrative example on page 8.)



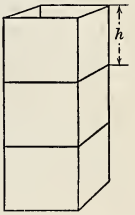
5. Write the following in short form: $t = l + l + l + l$.

6. What is the meaning of $t = 4\,l$?

7. The floor of a schoolroom was found to consist of nineteen boards, all of the same width. If the width of one board is called w, what is the total width of the floor?

8. If there are in the schoolroom four windows, of equal width, what is their combined width? (Use w.)

9. If three chalk boxes of the same height are placed on top of each other, and if the height of one box is called h, how would you express the total height of the three boxes?



10. If ten such boxes were placed on top of each other, how would you denote their total height?

11. What is the meaning of $t = 20\,h$?

3. Where Algebra is Used. Algebra is used in many trades, industries, and professions. Business men use algebra when they face a difficult problem in investment or insurance; engineers, architects, and contractors use algebra in solving "construction" problems; the scientist uses algebra for many questions that arise in his laboratory.

Oral Exercises. ***Simple Applications of Algebra***

1. $t = h + h =$ _(?)_.
2. $t = d + 2\,d =$ _(?)_.
3. $t = w + 3\,w =$ _(?)_.
4. $t = h + 2\,h + 4\,h =$ _(?)_.
5. $t = 6\,h + 8\,h + 10\,h =$ _(?)_.
6. $t = 4\frac{1}{2}\,w + w + \frac{1}{2}\,w =$ _(?)_.

7. What is the meaning of $4\,b$? $5\,c$? $6\,d$?

8. If n stands for any *number*, what is the meaning of $2\,n$? $3\,n$? $\frac{1}{2}\,n$?

9. If w stands for the *weight* of a box, what is the meaning of $10\,w$? $50\,w$? $\frac{1}{5}\,w$?

10. If p stands for the *profit* of a certain firm, what is the meaning of $6\,p$? $\frac{1}{4}\,p$? $1.9\,p$?

11. If d means a certain *distance*, how long is $2\,d$? $5\,d$? $2.5\,d$?

12. In 1913 the *rent* for a certain house was r; today it is $3\,r$. Explain.

13. Last year the *cost* of a pound of sugar was c. Today it is $1.1\,c$. Explain.

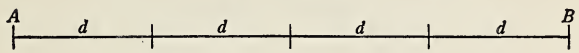
14. The *interest rate* of a bank during a certain month rose from r to $1.4\,r$. How much was the interest rate increased?

15. The noon *temperatures* of six cities on a certain day compared as follows: t; $1.1\,t$; $\frac{7}{8}\,t$; $1.2\,t$; $.85\,t$; $1.17\,t$. Explain.

4. Picturing Algebraic Shorthand. As you have seen, many letters of the alphabet may be used for the purposes of mathematical shorthand. The algebraic meaning of any particular letter depends on the situation in which it is used. The following exercises suggest how simple diagrams may help to make clear the meaning of algebraic shorthand.

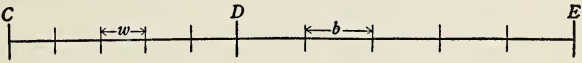
Exercises. *Picturing Algebraic Shorthand*

1. Suppose a train takes four hours to go from Calgary to Lethbridge, covering a distance of d miles each hour.



In the diagram, let A and B represent these two cities. Using the letter d, write the algebraic expression which represents the total distance from A to B. (The distance from A to B is often referred to as the distance AB.)

2. In the figure, how long is the distance CD? the distance DE?



3. A wire was bent into a zigzag design, as shown, all the parts of the zigzag being equal. How long was the wire?

4. If w represents the width of a house, illustrate by a diagram the width, $2w$, of another house.

5. If h represents the height of a tree, illustrate by a diagram the height of a tree which is $1\frac{1}{2}\,h$.

5. The Vocabulary of Algebra. Like every other language, algebra has its own word list, or *vocabulary*. Most of this vocabulary was adopted many years ago. Read and study the explanations of new words very carefully. Much of your progress will depend on a clear grasp of their meaning.

If you fail to give proper attention to the *vocabulary of algebra,* you will feel like a traveller in a strange land who has not learned the language of the people.

6. Some New Words Explained. The following new terms should be read and studied with great care.

Algebraic Expression. Any combination of numbers, letters denoting numbers, and signs of operation, when written according to the rules of algebra, is an *algebraic expression.*

Thus, $6\,h$, $h + h$, and $7\,w - 4$ are algebraic expressions.

Factors. The expression $6\,w$ is a *product,* and 6 and w are its *factors.* 6 is a *numerical* factor and w is a *literal* factor. The factors of a product, multiplied together, form the product.

Numerical Coefficient. In an expression such as $3\,w$, 3 is called the *numerical coefficient* of w, or simply the *coefficient* of w. It shows how many times the letter following it is to be used as an addend.

Thus, $4\,w = w + w + w + w$.

Remember that w has the same meaning as $1\,w$. Remember also that when there are two factors, one numerical and one literal, the numerical factor is written first and so becomes the numerical coefficient.

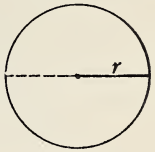
Test your knowledge of each of these terms by explaining, in your own words, the meaning of each.

Exercises. *Using Letters in Algebra*

In each of the following expressions point out (1) *the literal factor;* (2) *the numerical factor:*

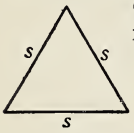
1. $3\,w$. 2. $5\,h$. 3. $2\,l$. 4. w. 5. $4\frac{1}{5}\,h$. 6. $7.4\,d$.

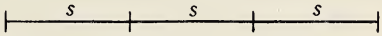
7. In the adjoining figure, r represents the radius of the circle. What expression represents the diameter of the circle?



8. If you picture w by a line 2 inches long, how long a line would be needed to picture $2\,w$? $5\,w$? $1\frac{1}{2}\,w$?

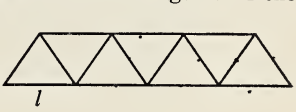
9. Place three equal pencils or rulers so as to form an *equilateral* triangle. Each of the rulers represents a side of the triangle.





If the three rulers are placed end to end in a straight line, we obtain a picture of the boundary, or the *perimeter*, of the triangle. Show that the perimeter formula of the equilateral triangle is $p = s + s + s$, or $p = 3\,s$.

10. In the steel frames of bridges or roofs you will often see sets of triangles such as are shown in the figure. If the frame is made of steel beams, each having the length l, find the total length of all the beams shown in the figure.



11. If the side of an equilateral triangle is $2\,d$, write its perimeter formula.

Solution. $p = 2\,d + 2\,d + 2\,d$, or $p = 6\,d$.

12. Write the perimeter formula for the six-pointed star shown in the figure, using s to represent the side of each equilateral triangle.

7. Similar Terms. You will now be able to understand the following explanations:

Term. Algebraic expressions, such as $3\,s$ and $2\,s$, may be combined by *addition* or by *subtraction.* In that case the expressions used in forming the *sum* or the *difference* are called *terms.*

Thus, when we write $6\,w + 4\,w$, the terms are $6\,w$ and $4\,w$.

Similar Terms. Two terms which have the same literal factor are said to be *similar.*

Thus, $5\,h$ and $6\,h$ are *similar terms.*

Dissimilar Terms. Two terms which do not have the same literal factor are said to be *dissimilar.*

Thus, $4\,a$ and $6\,b$ are *dissimilar terms.*

8. Combining Similar Terms. If the sides of two squares are denoted by w and $3\,w$, the total length of their perimeters is $4\,w + 12\,w$, or $16\,w$. The single term $16\,w$ is called the *sum* of the two original terms.

Moreover, in the case of these two squares, how much longer is the perimeter of the larger square? The answer is $12\,w - 4\,w$. This result may be expressed by the single term $8\,w$, which is called the *difference* of the two original terms.

From these exercises we may infer the following important

Rule for Combining Similar Terms

1. *The sum of similar terms* may be obtained by finding the *sum* of the coefficients of the given terms and multiplying it by the literal factor appearing in each of the terms.

Thus, $4\,a + 6\,a + 7\,a = 17\,a$.

2. *The difference of two similar terms* may be obtained by finding the *difference* between the coefficients of the given terms and multiplying it by the literal factor appearing in each of the terms.

Thus, $9a - 2a = 7a$.

NOTE. This rule will be called the *Rule of Similar Terms*, and the abbreviation S.T. may be used as a short way of referring to it.

Exercises. *Combining Similar Terms*

In each of the following exercises apply the Rule of Similar Terms:

1. Add:

	$20n$	$75b$	$91c$	$71x$	w
	$30n$	$19b$	$17c$	x	$59w$

2. Subtract:

	$30d$	$16w$	$19n$	$17.6x$	$7\frac{1}{2}s$
	$19d$	$10w$	n	$8.9x$	$2\frac{1}{2}s$

3. Find the *sum* of the terms in each column:

$4a$	$3b$	$9w$	$20h$	$1.1d$
$6a$	$7b$	$10w$	$40h$	$2.3d$
$7a$	$9b$	w	$50h$	$4.9d$

4. Find the *difference* of the terms in each case:

$90n$	$100d$	$7\frac{1}{2}c$	$.89e$	$207r$
$80n$	$59d$	$5\frac{1}{2}c$	$.76e$	r

Combine the similar terms in each of the following examples:

5. $4a + 6a + 15a + 21a$.
6. $25n + 75n - 16n$.
7. $\frac{1}{2}d + \frac{1}{3}d$.
8. $2\frac{1}{4}n + 7\frac{1}{2}n$.
9. $b + 2.3b + 4.5b$.
10. $29c + 31c + 40c$.

11. $8.2\,e + 9.3\,e + 14.6\,e$.

12. $19\,n + 30\,n + 2\,n$.

13. $20\frac{1}{2}\,r + 50\frac{1}{4}\,r - 30\frac{1}{3}\,r$.

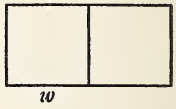
14. $50\,x - 24\,x + 5\,x$.

15. The profits of a certain business, during six consecutive years, were p, $\frac{1}{2}\,p$, $3\frac{1}{2}\,p$, $8.1\,p$, $2.3\,p$, and $9.2\,p$ respectively. Find the total profit.

16. In the preceding exercise by how much do the profits of the last three years exceed those of the first three years?

9. Evaluation of a Formula. The following example illustrates a very important process in the work with formulas:

Example. Each boy in a shop class was asked to make a wire model of a rectangular frame consisting of two equal square panels, of any convenient size, as shown in the figure.



Evidently, if the length of a side of one of the square panels were denoted by w, then the total length of the wire in the entire frame would be expressed by the formula

$$t = 7 \times w, \quad \text{or} \quad t = 7\,w. \quad \text{(Explain.)}$$

Now, before any particular pupil could tell how much wire he would need for his model, he had to decide how large a value he wished to use for w. Thus, if he decided to make $w = 2$ inches, he would need 14 inches of wire. To obtain this result, *he had to replace the letter* w, in the formula $t = 7 \times w$, *by the numerical value 2* (representing inches), thus obtaining $t = 7 \times 2$, or 14 inches.

The *table* at the top of the next page shows the values of w chosen by five boys of the class and the corresponding amounts of wire used by four of these boys. How many inches of wire did Charles use?

	Arthur	George	William	Richard	Charles
$w =$	3″	4″	5″	6″	10″
$t = 7\,w =$	21″	28″	35″	42″	(?)

Such a table is called a *substitution table,* since each value of t is obtained by *substituting,* in the expression $7\,w$, the corresponding value of w. That is, in each case we must multiply 7 by the given value of w.

This illustrative example will enable you to understand the following explanation:

Evaluation is the process of putting a number in place of a letter in a given algebraic expression, and thus finding the corresponding numerical value of the expression.

This process is often called *substitution,* for we are replacing a letter by a number. Observe that *the numerical value of an algebraic expression depends upon the numerical values which are substituted for the letters in the expression.*

Exercises. *Evaluating Formulas*

1. Copy the following table and fill in the values of p by substituting in the formula the given values of w. All the dimensions are given in inches.

If $w =$	4	5	7	9	12
then $p = 8\,w =$	32	(?)	(?)	(?)	(?)

2. Evaluate the formula $a = 9\,b$ for the given values of b.

$b =$	2	5	6.5	10.4	40.6
$a =$	(?)	(?)	(?)	(?)	(?)

3. Find the numerical values of t corresponding to the values given in the first column in these tables.

(1)

h	$t = 10\,h$
4	(?)
7	(?)
9.5	(?)
$10\frac{1}{2}$	(?)

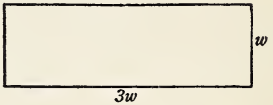
(2)

c	$t = 8\,c$
\$4.00	(?)
\$5.20	(?)
\$6.30	(?)
\$7.90	(?)

4. Write a formula for the perimeter of the adjoining rectangle. (Use p for perimeter.)

(1) If $w = 5$ in., find p.

(2) If $w = 2$ ft., find p.

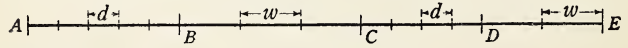


5. Given the formula $t = 25\,n$. Make a substitution table giving the values of t corresponding to the following values of n: 0, 1, 2, 5, 10, 15, 20.

6. Given the formula $p = 15\,w$. Make a substitution table showing the values of p for these successive values of w: 2, 5, 6, 8.2, $16\frac{2}{3}$.

10. Simplifying Formulas. Dissimilar Terms. There are many problems in algebra which require the use of two or more different letters. In all these cases we must know how to deal with *dissimilar terms.*

For example, in the following diagram, what is the total distance (t) from A to E?



Evidently, $t = 5\,d + 3\,w + 4\,d + 2\,w$.

This statement may be written in the form

$$t = 5\,d + 4\,d + 3\,w + 2\,w.$$

That is, $t = 9\,d + 5\,w$.

Observe that in the resulting formula *two* terms have replaced the *four* terms we had originally. To make this simpler form possible, we used not only the *Rule of Similar Terms* but two more rules which will now be explained.

11. Changing the Order of Terms. If you have to add 4, 5, and 6 you might say $4 + 5 + 6$, or $5 + 4 + 6$, or $6 + 4 + 5$, etc., but the final result would be the same in every case.

From this illustration we infer the following

Rule of Order for Addition

In addition the addends may be combined in any order.

Thus, if a and b are to be added, we have

$$a + b = b + a,$$

and if a, b, and c are to be added, we have

$$a + b + c = b + c + a = c + a + b = c + b + a, \text{ etc.}$$

12. Grouping of Terms. To find the sum of 4, 6, and 8, we may first combine 4 and 6 and then add 8, or we may add 4 to the sum of 6 and 8. In either case we form a *group* made up of two of the addends. That is,

$$4 + 6 + 8 = 4 + (6 + 8) = (4 + 6) + 8.$$

The symbols enclosing each of the groups are called *parentheses.*

Rule of Grouping for Addition

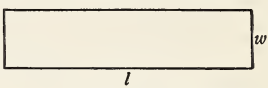
In addition the addends may be grouped in any manner.

Thus, $a + b + c = (a + b) + c = a + (b + c)$.

NOTE. The Rule of Order will sometimes be referred to by the capital letter O, and the Rule of Grouping by G. Parentheses were introduced into algebra by a Fleming, Albert Girard, in 1629. They may have a variety of forms. Thus, instead of the ordinary parentheses, one may use brackets [ ] or braces { } for the same purpose.

The importance and use of these two rules will become clear from the following example:

Example. A rectangular field is to be enclosed by a fence. If the dimensions, in feet, are l and w, how many feet of fence are necessary?



Evidently, $p = l + w + l + w$. Hence we have

$p = l + l + w + w$, [By the Rule of Order]

and therefore $p = (l + l) + (w + w)$. [By the Rule of Grouping]

Hence $p = 2\,l + 2\,w$. [By the Rule of Similar Terms]

This work may be condensed as follows:

$p = l + l + w + w$. [By O.]

$p = (l + l) + (w + w)$. [By G.]

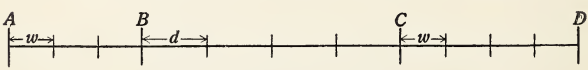
$p = 2\,l + 2\,w$. [By S.T.]

Exercises. *Dissimilar Terms*

Numbers 1 to 5, oral

In the following exercises apply the rules given above whenever it is possible:

1. Simplify the formula $t = a + b + 2\,a$.
2. Simplify the formula $t = 2\,d + 5\,r + 7\,d + 9\,r$.
3. Simplify the formula $s = 9\,n + 7\,c + 10\,n + 20\,c$.
4. Combine $4\,a + 5\,b + 6\,a + 7\,b$.
5. Find the length of AD in the following diagram



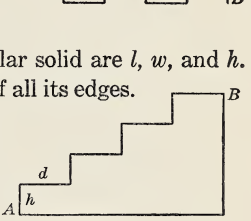
6. Three automobiles travel at hourly rates of d, l, and w respectively, all distances being expressed in miles. Find the total distance they cover in five hours.

7. The dimensions of a rectangle are $2\,w$ and $3\,h$. Find its perimeter p.

8. The accompanying design is called a *fret*. Observe how often w and h are repeated in the design. Find its total length from A to B.

9. The dimensions of a rectangular solid are l, w, and h. Find a formula for the total length of all its edges.

10. The figure shows the end view of a staircase. How long is a carpet covering this staircase from A to B?



NOTE. In evaluating a formula (1) substitute the given numerical values for the letters, (2) carry out the indicated multiplications or divisions, (3) add or subtract as required. For example, if $l = 5$ and $w = 3$, we might have $2\,l + 4\,w = 2 \times 5 + 4 \times 3 = 10 + 12 = 22$.

11. Using the formula $p = a + b + c$, find p when $a = 20$, $b = 24$, and $c = 30$.

12. Using the formula $t = 10\,a + 5\,b$, find t if $a = 9$ and $b = 4$.

13. If $a = 2$, $b = 3$, and $c = 4$, find the value of $8a + 7b - 3c$.

14. If $r = 10$, $s = 7$, and $t = 4\frac{1}{2}$, find the value of $5r - 7s + 8t$.

15. Combine $10\,a + 6\,a + a - 9\,a + 15\,a + 7\,a - 20\,a$.

13. The Fundamental Operations. You will recall that the fundamental operations of arithmetic are addition, subtraction, multiplication, and division. These operations are often denoted by the abbreviations A., S., M., D., respectively. Since they are also used constantly in algebra, a review of certain important terms is suggested at this point.

In the case of each of these processes the work may be arranged either *vertically* or *horizontally.*

Addition	Subtraction
(1) $8a$ Addend $2a$ Addend $10a$ Sum	(1) $8a$ Minuend $2a$ Subtrahend $6a$ Difference
(2) $8a + 2a = 10a$	(2) $8a - 2a = 6a$

Multiplication	Division
(1) $2a$ Multiplicand 7 Multiplier $14a$ Product	(1) $\frac{10a}{5} = 2a$ $10a$ Dividend 5 Divisor $2a$ Quotient
(2) $7 \times 2a = 14a$	(2) $10a \div 5 = 2a$

14. Different Forms. The expression $7 \times 2a$ is often written in the form $7 \cdot 2a$. Observe that the dot is placed somewhat higher than a decimal point in order to avoid confusion. The expression $\frac{10a}{5}$ may also be written $\frac{1}{5} \times 10a$.

The sum of a literal expression and an arithmetic number is indicated by writing a *plus* sign between them. Thus, if n is *increased* by 5, we have $n + 5$. Similarly, if n is *diminished* by 5, we have $n - 5$.

In the following exercises the fundamental operations appear in simple combinations which illustrate still further the use of letters and numbers in algebra.

Practice Exercises. *Fundamental Operations*

Try to complete all the exercises in 10 minutes. Write answers only in your notebook.

1. Add:	$7h$	$20d$	$90h$	$17.1x$	$4\frac{1}{2}y$
	$5h$	$70d$	h	$5x$	$7y$

2. Subtract:

$20\,a$	$17\,n$	$8\,x$	$19.7\,x$	$7\frac{1}{3}\,y$
$14\,a$	$14\,n$	x	$14.9\,x$	$2\,y$

Find the products which are indicated below:

3. $5 \times 6\,a$.	7. $10 \cdot 4\,x$.	11. $12 \cdot \frac{2}{3}\,h$.	15. $10 \times 5.1\,a$.
4. $9 \times 7\,b$.	8. $12 \cdot 5\,a$.	12. $16 \cdot \frac{3}{4}\,n$.	16. $100 \times 4.2\,x$.
5. $10 \times 14\,h$.	9. $20 \cdot 3\,a$.	13. $20 \cdot \frac{4}{5}\,x$.	17. $50 \times 4.5\,h$.
6. $8 \times 15\,d$.	10. $18 \cdot 2\,d$.	14. $60 \cdot \frac{5}{6}\,w$.	18. $10 \times 9.2\,h$.

Find the quotients which are indicated below:

19. $12\,a \div 3$.

20. $20\,w \div 4$.

21. $40\,x \div 5$.

22. $36\,n \div 12$.

23. $\frac{42\,h}{6}$.

24. $\frac{4.8\,n}{12}$.

25. $\frac{5.04\,x}{2}$.

26. $\frac{50\,n}{25}$.

27. $\frac{360\,n}{72}$.

28. $\frac{39\,x}{13}$.

29. Find $\frac{5}{6}$ of $60\,w$.

30. Divide $4.82\,h$ by .2.

31. $\frac{3}{4} \cdot 28\,y = ?$

32. $3\frac{1}{7} \cdot 21\,a = ?$

33. Combine $n + n + 1 + n + 2 + n + 3$.

34. Combine $2\,n + 2\,n + 2 + 2\,n + 4 + 2\,n + 6$.

35. Simplify $t = 8\,d + 7 - 4\,d + 6 + 2\,d - 5 + d$.

36. Simplify $t = n + 3 + 2\,n + 6 + 9\,n + 12$.

37. Simplify $t = w + 7 + 6\,w - 5 - 2\,w + 10$.

Exercises. *Using Changing or Related Values*

1. Suppose that the temperature this morning, in degrees, is represented by t. How should you denote the temperature now if in the meantime it has risen 5 degrees?

Solution. The temperature now may be indicated by the expression $t + 5$.

2. The present age of a boy, expressed in years, is a. What expression represents his age 1 year ago? 5 years from now?

3. If n represents any whole number, express algebraically a number five times as large; one fifth as large; 5 larger; 5 smaller.

4. The side of a square is $w + 2$. Find the perimeter (p) of the square.

Solution. $p = w + 2 + w + 2 + w + 2 + w + 2$.
Hence $p = w + w + w + w + 2 + 2 + 2 + 2$. (Explain.)
Then $p = (w + w + w + w) + (2 + 2 + 2 + 2)$. (Explain.)
That is, $p = 4w + 8$. (Explain.)

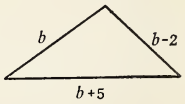
NOTE. The result obtained in Exercise 4 could have been found more quickly by a process of *multiplication*, as shown at the right. That is, $4 \times (w + 2)$, or $4(w + 2)$, equals $4w + 8$.

$$\begin{array}{r} w + 2 \\ \underline{4} \\ p = 4w + 8 \end{array}$$

5. The dimensions of a rectangle are w and $w + 6$. Find the perimeter of the rectangle.

6. The daily milages of a motorist, on four successive days, were m, $2m + 1$, $3m + 7$, and $5m - 1$. What was his total milage?

7. The sides of a triangle have the lengths b, $b - 2$, and $b + 5$ respectively. Write the formula for its perimeter (p).



8. Simplify the formula $p = 3s + 7 + 4s + 8$.

9. Simplify the formula $t = w + 14 + 6w - 1 + 9w$.

10. Simplify the formula $b = a + 10 + 3a - 5 + 8a$.

15. The Magic of the Formula. When a modern building is to be erected, every detail of the work must be planned before the actual construction can begin. The exact dimensions of every part must be indicated on the plans. Then a considerable amount of "figuring" must be done.

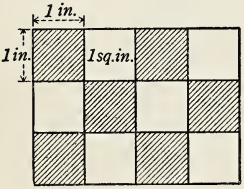
It is necessary to find out how much excavating is required and how much steel is needed for the framework. The size and strength of each steel beam must be determined.

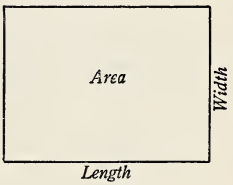
Much of this work is simplified by the use of *formulas*. With the aid of *mathematical formulas*, engineers, contractors, and business men solve such problems quickly and accurately, by a sort of magic.

In the following pages we shall deal with a few of the simplest formulas used in mensuration and business. Try to find out in each case how the particular formula that you are studying may be applied in actual life situations.

1. Formulas of Mensuration

16. Area of a Rectangle. If a rectangle is 4 inches long and 3 inches wide, its area is 4×3, or 12 square inches. Similarly, if the dimensions of a rectangle are expressed in





terms of the same unit (for example, inches, feet, yards, centimeters), *the area equals the product of the numbers which measure the length and the width.* In brief form, this rule may be written

$$\textit{Area} = \textit{length times width},$$

or $$A = l \times w,$$

or $$A = lw.$$

Observe that when two letters are written side by side in algebra, it means that we are to *multiply* together the numbers which they represent.

Illustrative Example. Find the area of a floor which is 20 feet long and 12 feet wide.

Solution. Using the formula $A = lw$, we replace l by 20 and w by 12. Hence we have $A = 20 \times 12 = 240$; that is, the area is 240 square feet. We have thus *evaluated* the formula $A = lw$.

NOTE. If the dimensions of a rectangle are 4 and 3, its area may obviously be found by writing either 4×3 or 3×4, the result being 12 in either case. That is, *the factors in a product may be written in any order*. Hence, the formula $A = lw$ may also be written $A = wl$.

Exercises. *Rectangles*

1. Using the formula $A = lw$, find A if $l = 24$ and $w = 14$.

Solution. Substituting 24 for l and 14 for w, we have

$$A = 24 \times 14, \quad \text{or} \quad A = 336.$$

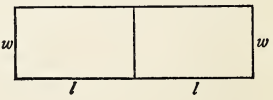
2. Using the formula $A = lw$, find A if

(1) $l = 20$ ft., $w = 10$ ft. (3) $l = 20$ in., $w = 6$ in.
(2) $l = 3$ yd., $w = 2$ yd. (4) $l = 14$ cm., $w = 5$ cm.

3. Using the formula $A = lw$, find the value of w if

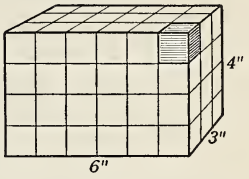
(1) $A = 20$, $l = 5$. (3) $l = 6$, $A = 12$.
(2) $A = 100$, $l = 25$. (4) $l = 8$, $A = 15$.

4. Place two equal rectangular sheets of paper side by side as shown in the accompanying figure. What is the total area of the new rectangle thus formed?



Solution. Evidently, $T = lw + lw$.
This may be written $T = 2\,lw$.

17. Volume of a Rectangular Solid. The figure represents a rectangular solid 6 inches long, 3 inches wide, and 4 inches high. We may imagine this box made up of layers of equal *cubes*, as shown in the figure. Evidently it contains $6 \times 3 \times 4$ of these cubes. Hence the volume is 72 cubic inches. *The volume of any rectangular solid is the product of the length, the width, and the height.* This rule may be expressed in short form as follows:



$$V = l \times w \times h,$$

or

$$V = lwh.$$

NOTE. Since the factors of a product may be multiplied in any order, the formula $V = lwh$ may also be written in other forms, such as $V = wlh$, or $V = hlw$.

Exercises. *Rectangular Solids*

Using the formula $V = lwh$, find the missing values in each of the following exercises:

1. $l = 5$ in., $w = 4$ in., $h = 3$ in.

Solution. Substituting 5 for l, 4 for w, and 3 for h, we have

$$V = 5 \times 4 \times 3, \quad \text{or} \quad V = 60 \text{ cubic inches.}$$

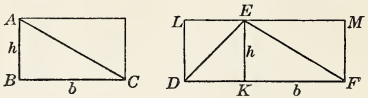
2. $l = 24$ ft., $w = 20$ ft., $h = 12$ ft.

3. $V = 70$ cu. ft., $l = 35$ ft., $w = 2$ ft.

4. $V = 160$ cu. ft., $w = 8$ ft., $l = 10$ ft.

5. Upon how many quantities does the volume of a rectangular solid depend?

18. Area of a Triangle. A study of the two adjoining figures will serve to make clear the following rule: *The area of a triangle is equal to half the product of its base and its height.* This rule may be shortened by means of the formula $A = \frac{1}{2}\,bh$, or $A = \dfrac{bh}{2}$.

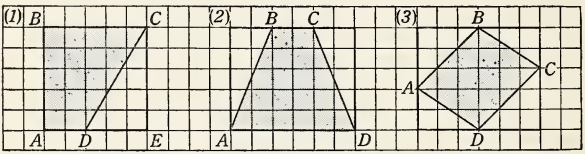


Exercises. *Triangles and Irregular Figures*

Using the formula $A = \frac{1}{2}\,bh$, find the missing number in each of the following exercises:

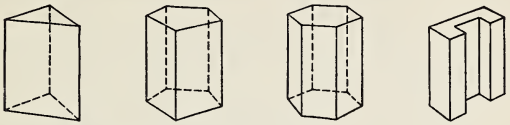
1. $b = 4$, $h = 6$.
2. $A = 20$, $b = 10$.
3. $A = 100$, $h = 20$.
4. $b = 20$, $h = 10$.
5. $b = 16\frac{1}{2}$, $h = 8$.
6. $A = 25$, $b = 8$.

7. By using squared paper, and applying the formulas for the area of a rectangle and of a triangle, it is often possible to find the areas of *irregular figures* such as the ones shown in the following diagrams:



Thus, in the first diagram, find the area of rectangle *ABCE*, and then subtract the area of triangle *CDE*. The result is the area of the irregular figure *ABCD*. Find the area of *ABCD* in each case.

19. Volume of a Prism. A *prism* has an upper base and a lower base. The figures below are all prisms.



It can easily be shown that

The volume of any prism may be found by multiplying the area of the base by the height.

This rule may be expressed by the formula

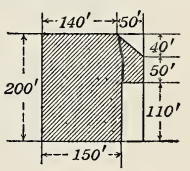
$$V = Bh.$$

Exercises. *Volumes of Prisms*

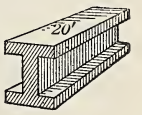
1. The base of a prism is a triangle whose area is 24. The altitude of the prism is 12. Find its volume.

2. The base of a prism has an area of 21 square feet. If the altitude of the prism is 10 feet, what is its volume?

3. The diagram shows the ground plan of a building. If the excavation for its foundation is 18 feet deep, how many cubic yards of earth were removed?



4. A steel beam of the form shown has a cross section containing 14.1 square inches. Its length is 20 feet. What is the weight of such a beam if steel weighs about 490 pounds per cubic foot? Show that the weight is approximately half a ton.

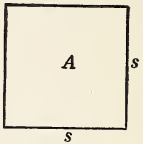


20. Area of a Square. The formula for the area of a square of side s is

$$A = s \times s,$$

or

$$A = s^2.$$



NOTE. The small figure 2 in the expression s^2 is called an *exponent*.

The way we write this product s^2 is a good example of an algebraic convention which must be respected. People might have agreed to write it otherwise, but we must follow the accepted rule. Distinguish carefully between $2s$ and s^2.

Exercises. *Squares*

1. What is the value of 4^2? 5^2? 7^2? 9^2? 10^2?

2. What is the meaning of b^2? of d^2?

3. Using the formula $A = s^2$, find s if $A = 64$.

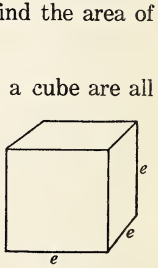
THINK: Which number, multiplied by itself, gives 64?

4. If $A = 144$; find s.

5. If $A = 400$; find s.

6. Could you use the formula $A = lw$ to find the area of a square?

21. Volume of a Cube. The dimensions of a cube are all equal. Hence the volume is found by multiplying together three equal numbers. Thus, if V is the volume of a cube (in cubic units), and e represents the length of one edge (in linear units), then $V = e \times e \times e$. This is abbreviated as follows:



$$V = e^3.$$

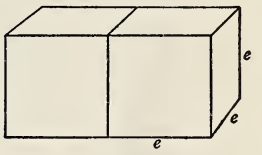
The expression e^3 is read "the cube of e," or "e cube."

NOTE. The small figure 3 in the expression e^3 is an *exponent*.

Exercises. *Cubes*

Find the volume of a cube which has an edge of

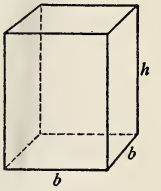
1. 5 in. 2. 2 ft. 3. 3 yd. 4. 3 cm.



5. How many cubic feet are there in 3 cubic yards?

6. If two cubes, each of edge e, are placed side by side as shown in the figure, what is the volume of the resulting rectangular solid?

7. By means of models, or by a drawing, explain the meaning of $4\,e^3$.



8. The volume of the solid shown in the accompanying figure is $b \times b \times h$. This may be written b^2h. What is the meaning of $2\,b^2h$?

Practice Exercises. *Fundamental Operations*

Try to complete all the exercises in 12 minutes. Write the answers only in your notebook.

1. *Add:*

$4\,x$	$9\,ab$	$30\,cde$	$4.7\,a^2$	$8\,n^3$
$7\,x$	$7\,ab$	$41\,cde$	$9.2\,a^2$	$7\,n^3$
$5\,x$	$10\,ab$	$59\,cde$	$10.5\,a^2$	$4\,n^3$

2. *Subtract:*

$91\,w$	$31\,a^2b$	$47\,ab^2$	$50\,x^3$	$19\,y^2$
$42\,w$	$19\,a^2b$	$29\,ab^2$	$21\,x^3$	y^2

Multiply:

3. $10 \times 7 \times 3\,a$.
4. $2 \times 5 \times 2\,w$.
5. $8 \cdot 2\,cd$.
6. $12 \times 5\,a^2$.
7. $6 \times 5\,x^3$.
8. $8 \cdot 7\,a^2b$.

Divide:

9. $\frac{7w}{7}$. 10. $\frac{8a}{8}$. 11. $\frac{30n}{5}$. 12. $\frac{42x^2}{7}$. 13. $\frac{60x^3}{12}$. 14. $\frac{4.5n^2}{3}$.

15. Combine $9ab + 7ac - 4ab + 4ac + 10ab - ac$.

16. Combine $7x^2 + 9y^2 - 5x^2 + 3y^2 + 6x^2 - 8y^2$.

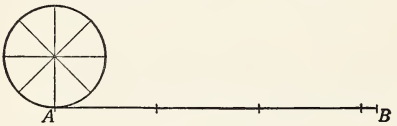
17. Combine $6a^3 + 14a^2b + 9ab^2 + 7a^2b - 6ab^2 - 3a^3$.

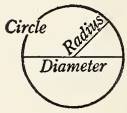
Observe that a^2b and ab^2 are *dissimilar* terms. Explain.

18. If $s = 7x^2 - 3y^2$, find s when (1) $x = 3$, $y = 4$; (2) $x = 10$, $y = 5$; (3) $x = 6$, $y = 2$; (4) $x = 9$, $y = 7$.

NOTE. (1) Find the squares of the numbers substituted for the letters, (2) perform the indicated multiplications, and (3) subtract as required.

22. Circumference of a Circle. To measure the circumference of any circular object, you may wind a sheet of paper around the object. It is then possible to mark off on the





paper the desired circumference. You will find that the circumference is $3\frac{1}{7}$ times the diameter. The formula is

$$C = 3\tfrac{1}{7} \times d.$$

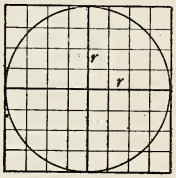
23. The Meaning of π (Pi). The value $3\frac{1}{7}$ is only approximate; more accurately it is 3.1416. Since no fraction or decimal gives the exact value to be used in the circumference formula, it is customary to represent this number by the Greek letter π. Hence the above formula is often stated thus: $C = \pi d$.

If r represents the radius of the circle, the formula becomes

$$C = 2\pi r.$$

24. Area of a Circle. Suppose that a circle of radius r is drawn on squared paper, as shown in the diagram. The large square in which the circle is enclosed may then be divided into four squares, each having a side equal to r. The area of the large square is readily seen to be $4r^2$. The area of the circle is therefore less than $4r^2$. It can be shown that the area of the circle is $3\frac{1}{7} \times r^2$, approximately. The exact formula is

$$A = \pi r^2.$$



Exercises. *Circles*

1. Find the circumference of a circle with a diameter of 7 inches.

Solution. We have $C = \pi d = \frac{22}{7} \times 7 = 22$; that is, the circumference is 22 inches.

Find the circumference of a circle having a diameter of

2. 21 in. 3. 6.1 ft. 4. 3.4 cm.

5. What is the area of a circle which has a radius of 3 inches?

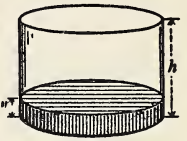
Solution. $A = \pi r^2 = \frac{22}{7} \times 3 \times 3 = \frac{198}{7} = 28.28 \cdots$; that is, to the nearest tenth of a square inch, the area is 28.3 square inches.

Find the area of a circle which has a radius of

6. 7 in. 7. 8 yd. 8. 8.4 in. 9. 9 mm.

10. Suppose that you have three circles with the same centre, but with radii of 1 inch, 2 inches, 3 inches. From a study of the formula $A = \pi r^2$, compare their areas.

25. Volume of a Cylinder. It is evident that if the base of a cylinder contains 4 square inches, and its height is 1 inch, its volume is 4×1 cubic inches. If its height is 2 inches, the volume is 4×2 cubic inches, and so on. Hence, as in the case of a prism, we may *find the volume of a cylinder by multiplying the area of its base by its height.*



If the radius of the cylinder is r, the area of its base is πr^2. Hence the formula for the volume of a circular cylinder becomes

$$V = \pi r^2 h.$$

Illustrative Example. A cylindrical glass jar has a radius of $3\frac{1}{2}$ inches. How much water will it contain when it is filled to a height of 6 inches?

Solution. $V = \pi r^2 h = \frac{\overset{11}{\cancel{22}}}{\cancel{7}} \times \frac{\cancel{7}}{\cancel{2}} \times \frac{7}{\cancel{2}} \times \frac{\overset{3}{\cancel{6}}}{\cancel{1}} = 231$ (cubic inches).

Exercises. *Cylinders*

1. A cylindrical hot-water boiler was 14 inches in diameter and 6 feet high. How many gallons of water did it hold? (One gallon contains 277 cubic inches.)

2. Using the formula $V = \pi r^2 h$, find the required values of V in the following table:

	r	h	V
(1)	7 in.	10 in.	
(2)	14 ft.	9 ft.	
(3)	20 ft.	8 ft.	
(4)	$2\frac{1}{2}$ in.	9 in.	
(5)	5 cm.	10 cm.	

2. The Use of the Formula in Business

26. The Simple-Interest Formula. In your arithmetic you learned that *interest* = *principal* × *rate* × *time*, or that

$$i = prt.$$

Illustrative Example. What is the interest on \$600, for 2 years, at 6%?

Solution. In this case $r = \frac{6}{100}$, $t = 2$, $p = 600$. Using the formula $i = prt$, we have $i = \cancel{600}^{6} \times \frac{6}{\cancel{100}} \times 2 = 72$; that is, the interest is \$72.

Exercises. *Formulas Used in Business*

1. What is the interest on \$700, for 3 years, at 3%?

2. What is the interest on \$400, for 2 years, at $5\frac{1}{2}$%?

Suggestion. Observe that $5\frac{1}{2}\% = \frac{11}{200}$.

3. Using the formula $i = prt$, find i in the following cases:

Interest	Principal	Rate	Time in Years
(1) ___	\$720	6%	$3\frac{1}{2}$
(2) ___	\$950	5%	$2\frac{1}{3}$
(3) ___	\$1040	$3\frac{1}{2}$%	$1\frac{1}{4}$
(4) ___	\$2080	$4\frac{1}{2}$%	$\frac{3}{4}$
(5) ___	\$860	$2\frac{1}{2}$%	$\frac{1}{4}$

4. The *amount formula* is $A = p + prt$; that is, *the amount equals the principal plus the interest.* Find A if $p = \$2500$, $r = 4\%$, and $t = 3$.

5. A man bought a \$10,000 home. He paid \$2000 in cash. For the balance he gave a mortgage, the interest rate being 6%. How much interest did he pay each year?

6. The *profit formula* is $p=s-c$; that is, *the profit equals the selling price minus the cost.* Find p if $s=\$3.50$ and $c=\$1.40$.

7. The distance travelled equals *rate* (miles per hour) times *time* (hours). Write a formula expressing this rule.

3. The Use of the Formula in Laboratories and Shops

27. Formulas Used by Scientists and Engineers. Scientists and engineers use a great many formulas, which help to simplify their work. Many of these formulas are quite complicated, but here are some exercises based on a few of the simpler ones.

Exercises. *Formulas Used in Science*

Find the value of the letter indicated in each case by substituting the given values in the formula:

1. *Pulley* formula: $DS = ds$.
If $D = 5\frac{1}{2}$, $S = 400$, and $d = 26$, find s.

2. Formula from *electricity*: $A = \frac{V}{R}$.
If $V = 110$, and $R = 440$, find A.

3. *Safe load* formula for steel beam: $W = \frac{1695\,AD}{L}$.
If $A = 5.34$, $D = 8$, and $L = 16$, find W.

4. *Pressure* formula: $P = ahd$.
If $a = \frac{1}{144}$, $h = 72$, and $d = 62.4$, find P.

5. Formula for *speed of cutting tool*: $n = \frac{3.8\,s}{d}$.
If $s = 40$, and $d = 5$, find n.

6. An *airplane* formula: $R = KSV^2$.
If $K = .000541$, $S = 36$, and $V = 92$, find R.

7. *Power shaft* formula: $M = \frac{D^3R}{125}$.
If $D = 2$, and $R = 600$, find M.

28. The Formula as an Expression of Relationships. Any formula of the kinds which have been considered thus far really represents *a shortened mathematical rule.* It may also be said that such a formula is a condensed statement of a mathematical *relation,* showing how certain quantities which may change in value *depend* on each other.

For example, the familiar formula $A = \pi r^2$, for the area A of a circle with radius r, expresses the relation between the radius of a circle and its area and shows exactly how the area of the circle *depends* on the length of the radius.

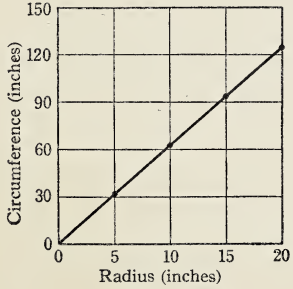
29. Three Ways of Expressing Relationships. Relationships such as that existing between the radius of a circle and its circumference may be expressed in three ways, namely, (1) by a *formula,* (2) by a *table,* and (3) by a *graph.* For example, taking the radius and the circumference of a circle, we have:

1. $C = 2\pi r$.

2.

Radius of circle (inches) =	5	10	15	20
Circumference of circle (inches) =	31.4	62.8	94.2	125.7

3.



The formula, the table, and the graph are further dealt with in section 71, and you will there learn more about their connection.

Exercises. *Expressing Relationships*

1. The following tables are based on the formula $c = 15\,g$. Find the missing values.

$g =$	6	8	11
$c = 15\,g =$	--	--	--

$c = 15\,g =$	\$.75	\$1.05	\$1.35
$g =$	--	--	--

2. Draw a graph of the formula $c = 15\,g$, using successive integral values * of g from 0 to 10.

3. The formula $F = \frac{9}{5}\,C + 32$ expresses the relationship between centigrade and Fahrenheit temperatures. Make a table and draw a graph using the values 0, 5, 10, 15, $\cdots$, 100 for C.

30. Obtaining Formulas from Verbal Statements. When a mathematical relation is given in verbal form, we may desire to express it algebraically. The following exercises will give practice in obtaining simple algebraic formulas which are based on verbal statements or descriptions.

Exercises. *From Words to Formulas*

1. To find the area of a rectangle we multiply the length by the width. Write the formula for this.

2. If c denotes the cost of one pound of goods and n is the number of pounds purchased, show that the *total cost* may be represented by the formula $T = nc$. Express the formula in words.

* Integral values of a variable are those represented by whole numbers, or integers, such as 1, 2, 3, and so on.

3. Write the formula you would use to find

(1) the area of the top of a round table;
(2) the area of the gable triangle of a house;
(3) the volume of a paint can;
(4) the cost of ribbon at so much per yard;
(5) the volume of a cubical water tank.

31. Obtaining Verbal Statements from Formulas. Sometimes you may see a formula which you wish to explain to someone who knows no algebra. The following exercises will give you practice in doing this.

Exercises. *From Formulas to Words*

Express in words each of the following formulas:

1. $A = \frac{bh}{2}$.
2. $i = prt$.
3. $V = \pi r^2 h$.
4. $A = s^2$.
5. $V = Bh$.
6. $C = 2\pi r$.

32. Summary. In this chapter you have learned

(1) The use of letters in algebra.
(2) The meaning of certain new words.
(3) The Rule of Similar Terms.
(4) The Rule of Order for Addition.
(5) The Rule of Grouping for Addition.
(6) How to evaluate formulas and algebraic expressions.
(7) How to combine letters and numbers by the four fundamental operations.
(8) Important formulas used in mensuration.
(9) The use of the formula in business and in science.
(10) Three ways of expressing relationships.
(11) How to obtain formulas from verbal statements.
(12) How to express formulas in words.

Test on Chapter I

1. Complete each of the following statements:

(1) In the term $7h$, the number 7 is called the __(?)__.

(2) Terms such as $10d$ and $4d$ are called __(?)__.

(3) The process of replacing a letter in an algebraic expression by a number is called __(?)__.

2. Combine the similar terms in the following:

(1) $d + d + d + d$.

(2) $3a + 6a + a$.

(3) $12\frac{1}{2}p - 3\frac{3}{4}p$.

(4) $7w + 9h + 3w + 6h$.

(5) $3.2d + 14.5d - .3d$.

(6) $6\frac{1}{2}c + 3\frac{1}{4}d + \frac{3}{4}d + 3\frac{1}{2}c$.

3. Find the *difference* of the terms in each case:

(1)	(2)	(3)	(4)	(5)
$8h$	$6c$	$10n$	$6\frac{3}{4}w$	$6.9r$
$3h$	c	$3\frac{1}{2}n$	$2\frac{1}{2}w$	$1.5r$

4. Find the *product* in each of the following:

(1)	(2)	(3)	(4)	(5)
$4d$	$3n$	$6a$	$9w \times 3$	$7 \times 4h$
2	8	$2\frac{1}{2}$		

5. Find the *quotient* in each of the following:

(1) $\frac{20b}{4}$. (2) $\frac{9c}{9}$. (3) $\frac{2.4n}{8}$. (4) $60w \div 12$. (5) $3.68p \div 4$

6. A man's profits for four consecutive years are represented by p, $1\frac{3}{4}p$, $1\frac{1}{2}p$, and $4\frac{1}{2}p$ respectively.

(1) Find the total profit for four years.

(2) How much more did he make during the fourth year than during the third?

7. Find the missing value in each of the following:

(1) $V = lwh$. If $l = 2$, $w = 5$, and $h = 3$, then $V =$ __(?)__.

(2) $A = s^2 - w^2$. If $s = 6$ and $w = 1$, then $A =$ __(?)__.

(3) $p = 2l + 2w$. If $l = 12$ and $w = 5$, then $p =$ __(?)__.

(4) $s = an$. If $s = 280$ and $n = 7$, then $a =$ __(?)__.

8. Find the missing numbers in each of the following:

(1) $V = lwh$.

$l =$	8 ft.	19 yd.
$w =$	3 ft.	12 yd.
$h =$	$\frac{1}{2}$ ft.	$\frac{3}{4}$ yd.
$V = lwh =$		

(2) $S = 6\,e^2$.

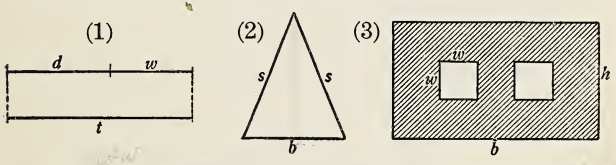
$e =$	5 in.	$\frac{2}{3}$ yd.
$S = 6\,e^2 =$		

9. (1) Write a formula for the wages (w) of a man who earns 85¢ an hour for h hours.

(2) Write a formula for the total money (m) of a boy who starts with 60¢ and saves 15¢ each week for w weeks.

(3) The cost (c) of sending a package by messenger in a certain city is 1¢ for each pound, plus 6¢. Write a formula for finding the cost for any number of pounds (p).

10. Write a formula suggested by each of the following:



$t = ?$ Perimeter (p) = ? Area (A) of shaded portion = ?

11. The length of a rectangle is 8″ and its width is w″.
(1) Its perimeter equals _(?)_. (2) Its area equals _(?)_.
(3) An area three times as large equals _(?)_.

12. Combine $20\,x^2y + 7\,xy^2 - x^2y + 10\,xy^2 + 5\,x^2y - 12\,xy^2$.

13. Combine $9.2\,ab + 7.3\,ac + 12 - 7\,ab + 2.7\,ac - 10$.

CHAPTER II

THE EQUATION

33. The Importance of Problems and Missing Numbers. In every numerical problem we are really trying to find the value of a missing number or of several missing numbers. There are many situations in everyday life which call for the finding of one or more missing numbers.

In the *home* it is of importance to know how much money can be set aside each week and month for food, clothing, and other necessary items. The *business man* and the *manufacturer* must find out their "overhead" expenses and know the cost of production and the probable margin of profit. The *builder* must determine the exact dimensions of all the parts of the house he is building, as well as the cost of each item in the contract. The *banker* must be able to tell what rate of interest he is obtaining from certain investments, and how to deal safely and profitably with deposits and other assets. The *engineer* must plan the dimensions and the necessary strength of the materials used in many structures, such as bridges, tunnels, skyscrapers, elevators, engines, motors, ships, and airplanes. The *government* must solve many financial problems of great importance to every citizen. *Inventors* and *scientists* are constantly trying to use our natural resources, as well as the great forces of heat, light, water, and electricity, for the improvement of living conditions. This requires the constant study of many problems.

Thus we see that not a person is living in the modern world whose welfare is not influenced very definitely by problems which someone must solve.

34. Solving Problems by the Equation Method. The simplest of these problems can be solved very quickly by ordinary arithmetic. When a problem is complicated, however, we usually need more powerful methods. The processes of algebra aid considerably in solving difficult problems. In particular, the great tool which algebra has furnished for finding missing numbers quickly and easily is the *equation method.* You will gradually become acquainted with this powerful method. We shall begin by comparing the way in which simple problems are solved by arithmetic and by algebra.

Illustrative Example 1. John's mother ordered five pounds of meat at the market. Her bill was 2 dollars. What was the price she had to pay per pound?

ARITHMETIC	ALGEBRA
Since five pounds cost 2 dollars, or 200 cents, and we wish to find the cost of one pound, we must divide 200 by 5.	Let c represent the cost of one pound, in cents.
Hence the cost of one pound is 40 cents.	Then $5c = 200$, and $c = 40$. (Why?)
	The statement $5c = 200$ is called an *equation.*

Illustrative Example 2. After travelling 4 hours in my automobile, I found that I had covered a distance of 84 miles. How fast had I travelled per hour?

ARITHMETIC. How do you solve this problem by arithmetic?

ALGEBRA. Let r represent the rate at which I travelled per hour.

Then $4r = 84$,

and $r = 21$. (Why?)

Again, the statement $4r = 84$ is called an *equation.*

Illustrative Example 3. James wishes to make a rectangular flower bed. He needs 40 square feet of land to plant the flowers he has bought. If the length of this plot is to be 8 feet, how wide must it be?

ARITHMETIC. How do you solve this problem by arithmetic?

ALGEBRA. Let w represent the width, in feet.

Then $8\,w = 40,$

and $w = 5.$ (Why?)

The statement $8\,w = 40$ is an *equation.*

Observe that the use of algebraic shorthand, in the solution of these problems, saves time and space. The following exercises will serve to give you some skill in writing equations which arise in the solution of simple problems.

Preliminary Exercises. *Making Equations*

In each problem represent the missing number by a letter. Then write the equation which may be used to solve the problem. Try to find the missing number in each case.

1. In 3 hours I covered a distance of 90 miles. What was my rate per hour? (Let r = the rate.)

2. If I can travel at the rate of 20 miles an hour, how many hours will it take me to reach a town which is 160 miles away? (Let h = the required number of hours.)

3. A grocer paid 32 dollars for 8 boxes of oranges. What was the cost per box? (Let c = the cost per box, in dollars.)

4. If butter costs 50 cents a pound, how many pounds can I buy for 2 dollars?

5. After working 40 hours, my wages were \$50. What was my hourly wage?

6. Ten years from now John's age will be 24 years. What is his present age?

7. Five years ago Henry was 8 years old. What is his present age?

8. If Carl's age were doubled, he would be 5 years older than his sister, who is 11 years old. How old is Carl?

9. A man left one half of his property to his wife. The remainder of the property was found to be $8000. What was the value of his property?

10. If each side of a square were made 1 foot longer, the perimeter would be 64 feet. How long is one side?

1. Solving Equations by Inspection

35. Finding the Missing Number. In every equation it is necessary to find the value of a missing number, or of some *unknown quantity*. When we have found the value of this missing number, we are said to have *solved* the equation. It is one of the principal objects of algebra to teach us how to solve problems by means of equations. The following exercises will give you some practice in solving equations of the simplest kind.

Exercises. *Solving Easy Equations*

1. What number must be added to 6 to make 10? State the number for which the question mark stands in $6 + ? = 10$.

2. Find the value of the missing number in each of the following equations:

$9 + (\ ) = 12.$ $\quad 7 + (\ ) = 30.$ $\quad (\ ) + 3.1 = 9.6.$
$8 + (\ ) = 20.$ $\quad (\ ) + 18 = 40.$ $\quad (\ ) + 10 = 100.$

3. In all the preceding exercises, how did you find the missing number?

4. State the number denoted by the question mark in each case:

$4 + ? = 13.$ $\quad ? + 14 = 26.$ $\quad 60 = 40 + ?.$

5. State the number denoted by the question mark in each case:

2 times $? = 18.$ $\quad \frac{1}{2}$ of $? = 6.$ $\quad \frac{3}{5}$ of $? = 9.$

4 times $? = 40.$ $\quad \frac{1}{3}$ of $? = 5.$ $\quad \frac{4}{5}$ of $? = 8.$

6. Find the value of the missing number in each case:

$4(\ \ ) = 12.$ $\quad 2(\ \ ) = 7.$

$5(\ \ ) = 15.$ $\quad 3.1(\ \ ) = 9.3.$

7. Find the value of n in each case:

$n + 9 = 12.$ $\quad n - 4 = 5.$ $\quad \frac{1}{3}n = 2.$

$6n = 18.$ $\quad n - 6 = 7.$ $\quad \frac{1}{4}n = 10.$

8. In each of these tables find in the first column the numbers corresponding to those given in the second column:

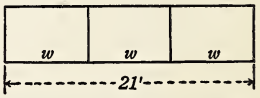
n	$n+2$
—	7
—	8
—	12
—	40
—	9.7

n	$3n$
—	6
—	9
—	21
—	60
—	9.3

n	$n-1$
—	7
—	9
—	15
—	400
—	9.3

n	$\frac{n}{4}$
—	2
—	5
—	10
—	80
—	2.3

9. Three boxes of equal widths were placed side by side, as shown in the figure. Find the value of w.



EXPLANATION. Since the three equal widths make up the total distance of 21 feet, we have the following equation:

$$3w = 21.$$

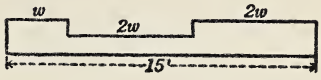
Hence $\quad w = 7.$ (Why?)

10. In the diagram, find the value of w and of $2w$.

Equation. $5w = 15$.

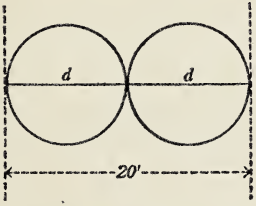
Solution. $w =$ _____.

$2w =$ _____.

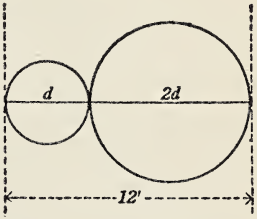


Find the diameters of these circles:

11.

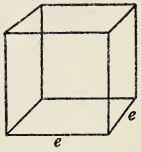


12.



36. Comparing Evaluation and the Equation Method.* There is a very interesting and important relation between problems that are solved by *evaluation* and those which are solved by means of an *equation*. Study the following illustrative examples:

Example 1. Each boy in a shop class was asked to make a wire model of a cube of any convenient size. A record was kept of the amount of wire each boy used for his model, as shown in the following table:



	JOHN	HENRY	JAMES	CARL	FRED
$e =$	1″	$1\frac{1}{2}$″	2″	3″	5″
$t = 12\,e =$	12″	18″	24″	36″	

How many inches of wire did Fred use? Observe that in each case the value of t is obtained by evaluating the formula $t = 12\,e$.

* While it is strongly recommended that this section be studied at this point, it may be postponed at the discretion of the teacher.

Example 2. Suppose that each boy in the shop class mentioned above had been given a definite length of wire and had been told to make a model of a cube. The question would then be: "*How large* a model can I make with *this* piece of wire?"

	JOHN	HENRY	JAMES	CARL	FRED
$t = 12\,e =$	24″	30″	36″	42″	60″
$e =$	2″				

Thus, if John received 24 inches of wire, he could make a two-inch cube, for if 12 equal pieces are necessary to make the model, *one* piece must be one twelfth of the total, or 2 inches. How large a model can each of the other boys make?

Observe that the missing value may be found in each case by solving an equation. Thus, in the case of John we have the equation

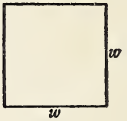
$$12\,e = 24,$$

from which we obtain $e = 2$.

You will see, then, that when a value of e was given, we found the corresponding value of t by *evaluation*. When a value of t was given, we found the corresponding value of e by *solving an equation*. In a certain sense, then, one process is the *reverse* of the other. The following exercises will serve to make you familiar with this important relation.

Exercises. *Evaluation and the Equation Method Compared*

1. A wire model of a square frame is to be made. Using the formula $p = 4\,w$, find the missing values in the following tables, all the dimensions being expressed in inches:



EVALUATION

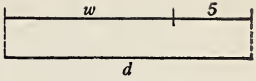
$w =$	3	5	9
$p = 4\,w =$			

EQUATION METHOD

$p = 4\,w =$	12	20	36
$w =$			

2. The accompanying diagram suggests the formula

$$d = w + 5.$$



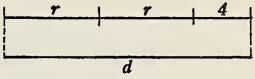
Using this formula, find the missing values in the following tables:

$w =$	2	10	16
$d = w + 5 =$			

$d = w + 5 =$	9	12	20
$w =$			

3. Write the formula suggested by the accompanying figure.

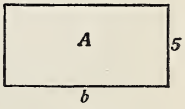
Using this formula, find the missing values in these tables:



$r =$	3	5	10
$d =$ ___ $=$			

$d =$ ___ $=$	12	20	64
$r =$			

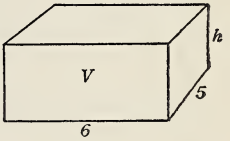
4. Using the formula based on the adjoining figure, find the missing values in the following tables, all the dimensions being in feet:



$b =$	8	10	12
$A = 5\,b =$			

$A = 5\,b =$	20	35	60
$b =$			

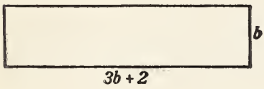
5. Write the formula for the volume of the rectangular solid shown in the diagram and thus find the missing values in the following tables:



$h =$	4	7	10
$V =$ ___ $=$			

$V =$ ___ $=$	60	90	100
$h =$			

6. Find the missing values in these tables, if p represents the perimeter of the figure below:



$b =$	4	7	10
$p =$ ___ $=$			

$p =$ ___ $=$	20	36	60
$b =$			
$3\,b + 2 =$			

2. Solving Equations by Rule

37. Importance of Skill in Solving Equations. Since equations are of very great assistance in solving mathematical problems, it is necessary for every student of algebra to develop skill in the use of this equation method.

In the following pages you will study certain explanations and rules which will enable you to solve equations quickly and accurately.

38. Equality. *An equality is a statement showing that two expressions stand for the same number.* Thus, the following statements represent different types of equalities.

1. $8 = 8$.
2. $4 + 2 = 6$.
3. $4\,w = 12$.
4. $7\,x + 2 = 9$.
5. $a + b = c$.
6. $A = lw$.

39. Identity. *An identity is an equality that states a fact.*

Thus, $4 + 2 = 6$ is an identity.

Similarly, $2\,n + 4\,n = 6\,n$ is an identity.

Also, $a + b = b + a$ is an identity.

For, whatever value is given to the letters which appear in either of these equalities, the statement remains an equality.

Hence, an equality which is true for all values of the letters which appear in it is an identity.

40. Equation. *An equation is an equality that asks a question.*

Thus, $5\,n + 2 = 17$ is an equation. It is really a short way of writing the question: "Five times *which* number, when two is added, will give the total result 17?"

In other words, this equation suggests a *particular* missing number which is to be found. Hence such an equality is true only if we replace the letter in the equation by the correct value.

Obviously, *it is the purpose of an equation to enable us to find quickly some missing number (or numbers) in a given problem.*

41. Root. The correct value of the missing number which is required in a given equation is called the *root* of the equation. The process of finding the root of an equation is often called the *solution* of the equation.

Thus, the solution of the equation $5\,n + 2 = 17$ shows that 3 is the root.

42. Checking a Solution. In order to make sure that the correct value of the missing number has been found, we must *check* each result. This is done by substituting the value found for the missing number in the given equation. If we obtain a true statement, or an identity, we know that the solution is correct.

Thus, the root of the equation $4\,w + 5 = 17$ is 3, for by substitution we find that

$$4 \times 3 + 5 = 17,$$

from which we have $12 + 5 = 17,$

or $17 = 17.$

The correct value of the root is said to *satisfy* the equation.

There are equations which have *more* than one root. For example, the equation $x^2 - 5\,x + 6 = 0$ has two roots, 2 and 3, as may be seen by substituting each of these values for x.

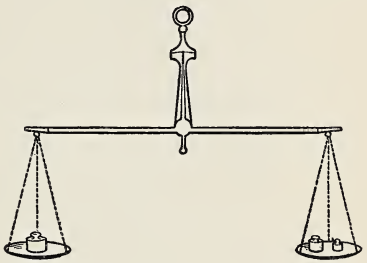
Practice Exercises. *Checking Solutions*

Copy the numbers of the following examples. In the case of each equation, check by substitution the indicated root. If the root checks, write the word "True"; if not, write "False."

1. $4a = 60.$	$a = 15.$
2. $7w + 1 = 15.$	$w = 2.$
3. $9h - 1 = 26.$	$h = 4.$
4. $4n + 7 - 2n = 15.$	$n = 2.$
5. $7x + 4 = 2x + 9.$	$x = 1.$
6. $\frac{2}{3}n + 6 = 10.$	$n = 6.$
7. $9 - 2a = 3.$	$a = 3.$
8. $25 - 3h = 40 - 6h.$	$h = 5.$
9. $4n + 1 = 17.$	$n = 6.$
10. $3d - 5 = 7.$	$d = 2.$
11. $a - 5 = 8.$	$a = 13.$
12. $x + 10 = 24.$	$x = 6.$
13. $40r - 10 = 7r + 30.$	$r = 4.$
14. $6 - a = 12 - 2a.$	$a = 4.$
15. $7c - 5 = 4c + 4.$	$c = 3.$

43. Axioms and the Rule of Equality. In the preceding pages you have solved equations by *inspection*. But just as your work in arithmetic was made more definite by learning some important rules, you will find it very helpful to solve equations by applying a few simple principles, often called *axioms*. A knowledge of these will assist greatly in planning the solution of equations and in obtaining the required results more rapidly.

The two expressions which are written on both sides of the equality sign in an equation are often called its *members*. The left side of an equation is called its *first member*, while the right side is called the *second member*. The two members of an equation may be compared to equal weights which are placed in the scale pans of an ordinary balance. So long as these loads remain equal, they balance each other. If you change one of these weights, you must make a corresponding change in the other weight. Otherwise there will no longer be a balance. Keeping this illustration in mind, you will readily be able to understand the following four numerical statements:



	Addition	Subtraction	Multiplication	Division
Given Equality:	$8 = 8$	$8 = 8$	$8 = 8$	$8 = 8$
	$2 = 2$	$2 = 2$	$2 = 2$	$2 = 2$
Resulting Equality:	$10 = 10$	$6 = 6$	$16 = 16$	$4 = 4$

Thus, if you have two equal numbers, as is the case in the statement $8 = 8$, you can *add* 2 to both numbers; or you can *subtract* 2 from both numbers; or you can *multiply* both numbers by 2; or you can *divide* both numbers by 2. In each case you obtain another equality, such as $10 = 10$, or $6 = 6$, and so on.

For example, suppose that each of two boys deposits 8 dollars in the school bank, and that later each withdraws 2 dollars. Then each boy has 6 dollars left in the bank. Similarly, it is easy to suggest suitable illustrations for the other three statements.

Hence we have the following four simple principles, or *axioms*:

1. ***Axiom of Addition.*** *If equals are added to equals, the sums are equal.*
2. ***Axiom of Subtraction.*** *If equals are subtracted from equals, the differences are equal.*
3. ***Axiom of Multiplication.*** *If equals are multiplied by equals, the products are equal.*
4. ***Axiom of Division.**** *If equals are divided by equals, the quotients are equal.*

These four statements may be combined into a single statement, called the Rule of Equality:

If equals are increased, or decreased, or multiplied, or divided by equals, the results are equal.

This rule is sometimes stated in the following form: *Whatever you do to one side of an equation you must also do to the other side.*

We shall now use the Rule of Equality in solving various types of equations.

44. Equations of the First Type. (Using Division.) Study the following illustrative examples:

Illustrative Example 1. Solve the equation $5n = 20$.

The given equation tells us that a certain number, when multiplied by 5, gives 20.

Solution.	(1) $5n = 20$	Given equation.
	(2) $5 = 5$	Identity.
	(3) $n = 4$	Axiom 4.

Here we have divided each member by 5, thus applying Axiom 4. In brief form, the solution may be written:

(1) $5n = 20$.	Given.
(2) $n = 4$.	D_5.

The abbreviation D_5 means that we have divided each member by 5.

* The case of division by zero is excluded, for reasons which will be considered in a later chapter.

Check. The answer is 4, for if we substitute 4 for n in the given equation we have

$$5 \times 4 = 20,$$

or

$$20 = 20.$$

Illustrative Example 2. Solve the equation $4.1\,a = 82$.

Solution.	(1) $4.1\,a = 82.$	Given.
	(2) $a = 20.$	$D_{4.1}$.
Check.	$4.1 \times 20 = 82,$	$\begin{array}{r} 4.1 \\ \underline{20} \\ 82.0 \end{array}$
or	$82 = 82.$	

Exercises. *Equations of the First Type*

Numbers 1 to 10, oral

Solve the following equations and check each result:

1. $4\,a = 12.$	8. $5\,d = 600.$	15. $4\,a = 5.$
2. $7\,c = 21.$	9. $8\,r = 17.$	16. $7\,w = 0.$
3. $10\,d = 100.$	10. $2\,p = 21.$	17. $3\,b = 2.$
4. $20\,h = 90.$	11. $4\,a + 3\,a = 21.$	18. $1 = 2\,d.$
5. $9\,x = 9.$	12. $7\,d - 2\,d = 40.$	19. $5\,x = 4.$
6. $12\,n = 84.$	13. $20\,r - 4\,r = 32.$	20. $6\,h = 2.4.$
7. $6\,p = 1200.$	14. $10\,w + w = 44.$	21. $2.3\,x = 69.$

Study each of the following problems. Find out what is required. Try to state the problem in the form of an equation, and solve it. + Check.

22. In 5 hours an automobile covered a distance of 150 miles. What was its average rate per hour?

SUGGESTION. If the average rate is called r, we have $5\,r = 150$. Explain.

23. If the perimeter of a square is to be 60 feet, how long must one side be?

24. A salesman found at the end of a year that his total commission exactly equalled his regular salary. His total income was \$6400. What was his salary? (Let $c =$ the commission. Then $2c = 6400$. Explain.)

25. In 10 years a man saved \$8970. How much did he save annually?

26. If you wish to reach in 5 hours a city which is 120 miles from your home, at what rate must you travel?

27. Two business partners wished to divide a profit of \$12,000. If one contributed twice as much capital as the other, how much should each receive?

28. There were 100 shares in a business owned by three partners. A owned 52 shares, B owned 30 shares, and C owned the rest. A profit of \$8400 is to be divided among them. How much should each receive?

29. A liter contains 61 cubic inches, very nearly. How many liters are there in 10 gallons, if one gallon contains 277 cubic inches?

Let l represent the number of liters. Then $61\,l = 10 \times 277$. Explain.

30. The light which leaves the sun reaches the earth in about $8\frac{1}{3}$ minutes. The sun's distance from the earth is about 93,000,000 miles. How far does light travel in a second?

45. Equations of the Second Type. (Using Multiplication.) Study the following illustrative example:

Illustrative Example. Solve the equation $\frac{n}{3} = 6$.

The given equation states that one third of a certain number is 6.

Solution.		
	(1) $\frac{n}{3} = 6$	Given equation.
	(2) $3 = 3$	Identity.
	(3) $n = 18$	Axiom 3.

Here we have multiplied each member of the given equation by 3, thus applying Axiom 3.

In brief form, this may be written:

(1) $\frac{n}{3} = 6.$	Given.
(2) $n = 18.$	M_3.

The abbreviation M_3 means that we have multiplied each member by 3.

NOTE. Remember that $\frac{n}{3}$ has the same meaning as $\frac{1}{3}\,n$.

Check. $\frac{1}{3}\,18 = 6,$

or $6 = 6.$

Exercises. *Equations of the Second Type*

Numbers 1 to 10, oral

Solve the following equations and check each result:

1. $\frac{n}{2} = 7.$
2. $\frac{d}{3} = 9.$
3. $\frac{x}{10} = 2.$
4. $\frac{a}{5} = 20.$
5. $50 = \frac{l}{10}.$
6. $19 = \frac{1}{4}\,x.$
7. $2 = \frac{1}{9}\,h.$
8. $\frac{1}{100}\,c = 3.$
9. $\frac{1}{100}\,p = 21.$
10. $\frac{1}{7}\,n = 1000.$
11. $\frac{n}{5} = 9.$
12. $\frac{n}{19} = 20.$
13. $100 = \frac{x}{7}.$
14. $\frac{x}{6} = 0.$
15. $\frac{a}{10} = 1\frac{1}{2}.$
16. $6.4 = \frac{h}{3}.$
17. $\frac{p}{100} = 4.72.$
18. $\frac{p}{100} = 17.93.$
19. $2\frac{1}{5} = \frac{h}{5}.$
20. $\frac{h}{8} = 5\frac{3}{4}.$
21. $\frac{x}{8} = 2\frac{1}{3}.$

Solve each of the following problems by means of an equation:

22. I decided to save one fifth of my income during a certain month. My savings during that month were $18.20. What was my income?

23. In a game Henry scored 21 points, which were one fourth of all the points scored. What was the total score in the game? 84

24. James is one fifth as old as his father. James is 8 years old. What is his father's age? 40

25. After completing one half of her lesson, Martha had worked 10 examples. How many examples had been assigned? 20

26. After reading 30 pages of a book, I found that I had read only one tenth of the total number of pages. How many pages were there in the book? 300

46. Equations of the Third Type. (Using Subtraction.) Study the following illustrative examples:

Illustrative Example 1. Solve the equation $n + 3 = 17$.

Solution.

(1)	$n + 3 = 17$	Given equation.
(2)	$3 = 3$	Identity.
(3)	$n = 14$	Axiom 2.

Here we have subtracted 3 from each member of the original equation, thus applying Axiom 2.

In brief form, this solution may be written:

(1)	$n + 3 = 17.$	Given.
(2)	$n = 14.$	S_3.

S_3 means that 3 was subtracted from each member of the original equation.

Check. $14 + 3 = 17,$

or $17 = 17.$

Illustrative Example 2. Solve the equation $x + 4.1 = 9.7$.

Solution.

(1)	$x + 4.1 = 9.7.$	Given.
(2)	$x = 5.6.$	$S_{4.1}$.

Check. $5.6 + 4.1 = 9.7,$

or $9.7 = 9.7.$

Exercises. *Equations of the Third Type*

Numbers 1 to 10, oral

Solve the following equations and check each result:

1. $a + 6 = 10$.
2. $n + 9 = 20$.
3. $w + 100 = 300$.
4. $d + 90 = 91$.
5. $n + 7.1 = 9.1$.
6. $x + 1\frac{1}{2} = 6\frac{1}{2}$.
7. $10 + x = 19$.
8. $20 + d = 60$.
9. $50 + n = 52$.
10. $100 + x = 101$.
11. $n + 4 = 6$.
12. $d + 10 = 17\frac{1}{2}$.
13. $x + \frac{1}{2} = 7\frac{1}{2}$.
14. $a + .1 = 9.1$.
15. $h + 17 = 30$.
16. $b + 1\frac{1}{3} = 6\frac{1}{3}$.
17. $c + 3.2 = 9.2$.
18. $x + 500 = 600$.
19. $t + 15\frac{1}{3} = 17$.
20. $r + 7.4 = 10$.

Study the following problems. Find out in each case what is required. Write the equation which represents the problem, and solve it.

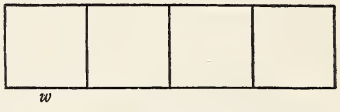
21. After receiving a salary increase of \$110, a clerk's salary was \$1400. What was the original salary?

22. A firm announced an increase of 5¢ per pound in the price of beef. The new price was 40¢. What was the original price?

23. By increasing the speed of his car 10 miles per hour, a chauffeur was able to cover a distance of 35 miles in one hour. What was his original speed?

24. The total length of wire needed to make a certain wire model of a cube was found to be 48 inches. How long was the edge of the cube?

25. A wire frame consisting of four equal panels, as shown in the diagram, required 52 inches of wire. How long was one side of each square?



26. The hourly rates of three trains, in miles, are d, $2d$, and $3d$ respectively. If the total distance covered by the trains in 4 hours was 480 miles, what was the rate of each train?

27. The annual incomes of four members of a family are i, $3i$, $4i$, and $10i$ respectively. Their combined annual income was found to be \$5400. How much did each of the four members earn per year?

28. Three men, A, B, and C, go into business together. B contributes twice as much capital as A, and C contributes three times as much capital as A. If the profit during a certain year was \$12,000, what was each man's share?

47. Equation of the Fourth Type. (Using Addition.) Study the following illustrative example:

Illustrative Example. Solve the equation $n - 4 = 5$.

Solution. This equation tells us that a certain number, when diminished by 4, gives 5.

(Remember that $10 - 4 + 4 = 10$. Similarly, $n - 4 + 4 = n$.)

(1)	$n - 4 = 5$	Given equation.
(2)	$4 = 4$	Identity.
(3)	$n = 9$	Axiom 1.

Here we have added 4 to each member of the original equation, thus applying Axiom 1. In brief form, this solution may be written:

(1)	$n - 4 = 5$.	Given.
(2)	$n = 9$.	A_4.

(What is the meaning of A_4?)

Check. $9 - 4 = 5$,

or $5 = 5$.

Exercises. ***Equations of the Fourth Type***

Numbers 1 to 10, oral

Solve the following equations and check each result:

1. $a - 2 = 7$.
2. $d - 7 = 10$.
3. $w - 6 = 100$.
4. $n - 17 = 1$.
5. $h - 6 = 49$.
6. $x - \frac{1}{2} = 7$.
7. $c - 1\frac{1}{3} = 2$.
8. $n - 1.1 = 9$.
9. $r - 2 = 7\frac{1}{2}$.
10. $x - 10 = 14.5$.
11. $x - 4 = 6$.
12. $n - 10 = 20$.
13. $d - 12 = 8$.
14. $w - 20 = 20$.
15. $x - 2\frac{1}{2} = 4\frac{1}{2}$.
16. $d - 3.9 = 6.1$.
17. $w - 100 = 0$.
18. $n - 5\frac{2}{3} = 8\frac{2}{3}$.
19. $r - 4\frac{5}{8} = 3\frac{7}{8}$.
20. $n - 8.7 = 15.2$.

Try to solve each of the following problems by the equation method:

21. I am thinking of a number which when diminished by 8 is equal to 20. What is the number?

22. After paying \$2.50, I still had \$7. How many dollars did I have at first?

23. After cutting off a piece of ribbon 7 inches long, I had 29 inches of ribbon left. What was the original length of the ribbon?

24. What number, when decreased by $29\frac{4}{5}$, becomes $31\frac{1}{5}$?

25. After travelling a distance of 49.1 miles, I still had to go 23.9 miles to reach my destination. What was the total distance I had planned to cover?

48. Summary. We have now studied four simple types of equations. The solution of each requires a special process. Complete the following table:

The *addition type*, such as $n + 3 = 7$, is solved by ***subtraction.***
The *subtraction type*, such as $n - 2 = 6$, is solved by _____.
The *multiplication type*, such as $4n = 12$, is solved by _____.
The *division type*, such as $\frac{n}{4} = 2$, is solved by _____.

What relation exists, therefore, between the operation shown in the *given* equation and the operation which is used in *solving* it?

Timed Practice Test

Time, 2 minutes

1. $4a = 12$.
2. $7w = 21$.
3. $8h = 120$.
4. $5d = 600$.
5. $9x = 9$.
6. $10x = 20$.
7. $12x = 60$.
8. $15n = 0$.
9. $105p = 31{,}500$.
10. $6p = 1200$.
11. $2a = 40$.
12. $3b = 90$.
13. $4n = 28$.
14. $4.1s = 8.2$.
15. $33d = 132$.
16. $14n = 56$.
17. $9.3h = 27.9$.
18. $4w = 100$.
19. $500c = 15{,}000$.
20. $25n = 25$.

Timed Practice Test

Time, 2 minutes

1. $n + 4 = 9$.
2. $n + 7 = 12$.
3. $w + 4 = 6$.
4. $d + 10 = 20$.
5. $a + .1 = 9.1$.
6. $x + \frac{1}{2} = 7\frac{1}{2}$.
7. $h + 17 = 30$.
8. $b + 1\frac{1}{3} = 6\frac{1}{3}$.
9. $n + 3.2 = 9.2$.
10. $x + 500 = 600$.
11. $x + 15\frac{1}{3} = 17$.
12. $r + 7.4 = 10$.

13. $n + \frac{1}{3} = 2$. 15. $n + 7\frac{1}{3} = 20$. 17. $n + 3.15 = 8.45$.
14. $n + 2\frac{1}{2} = 7$. 16. $n + 2.1 = 9.1$. 18. $n + 500 = 1500$.
19. $n + 60\frac{1}{2} = 90\frac{1}{2}$. 20. $n + 18.5 = 20.6$.

Timed Practice Test

Time, 2 minutes

1. $n - 1 = 8$. 4. $n - 2\frac{1}{2} = 3\frac{1}{3}$. 7. $x - 12 = 8$.
2. $n - 2 = 10$. 5. $n - 4 = 6$. 8. $x - 20 = 20$.
3. $n - 6 = 4$. 6. $x - 10 = 0$. 9. $x - 100 = 1$.
10. $n - 97.43 = 146.27$. 16. $y - 1 = 1$.
11. $n - 200 = \frac{1}{3}$. 17. $n - \frac{1}{2} = \frac{1}{2}$.
12. $n - 6.1 = 8.4$. 18. $n - 9.4 = 3.5$.
13. $x - 3.9 = 6.1$. 19. $n - 16.4 = 2$.
14. $n - 2\frac{1}{5} = 7$. 20. $n - 200 = 300$.
15. $y - 3.1 = 9$. 21. $x - 400 = 310$.

49. Equations Solved by Using Two Processes. As a rule, more than one operation is required to solve the equations which arise in dealing with actual problems. We shall begin by considering equations which may be solved by using two operations, or two "steps." The explanations which follow may easily be extended to the case of equations requiring for their solution three or more steps. Study the following illustrative examples:

Illustrative Example 1. Solve the equation $4n + 5 = 17$. (State the equation in words before solving.)

Solution. Subtracting 5 from each of these equals, we have

$$4n = 12. \qquad \text{[By Axiom 2]}$$

Dividing each member by 4, we have

$$n = 3. \qquad \text{[By Axiom 4]}$$

In brief form, the solution may be written:

(1)	$4n + 5 = 17.$	Given.
(2)	$4n = 12.$	$S_5.$
(3)	$n = 3.$	$D_4.$

Check. By substituting this result in the original equation, we have

$$4 \times 3 + 5 = 17,$$

or $$12 + 5 = 17,$$

or $$17 = 17.$$

Illustrative Example 2. Solve the equation $4a - 3 = 9.$ (State the equation in words before solving.)

Solution.

(1)	$4a - 3 = 9.$	Given.
(2)	$4a = 12.$	$A_3.$
(3)	$a = 3.$	$D_4.$

Check.

$$4 \times 3 - 3 = 9,$$

or $$12 - 3 = 9,$$

or $$9 = 9.$$

Illustrative Example 3. Solve the equation $\frac{n}{5} + 2 = 6.$ (State the equation in words before solving.)

Solution.

(1)	$\frac{n}{5} + 2 = 6.$	Given.
(2)	$\frac{n}{5} = 4.$	$S_2.$
(3)	$n = 20.$	$M_5.$

Check.

$$\frac{20}{5} + 2 = 6,$$

or $$4 + 2 = 6,$$

or $$6 = 6.$$

Illustrative Example 4. Solve the equation $\frac{2}{3}n = 8.$ (State the equation in words before solving.)

Solution by Inspection. Since $\frac{2}{3}$ of a certain number is 8, $\frac{1}{3}$ of that number must be 4, and hence the required number is 3×4, or 12.

Solution by Rule.

FIRST METHOD.

(1)	$\frac{2}{3}n = 8.$	Given.
(2)	$\frac{1}{3}n = 4.$	$D_2.$
(3)	$n = 12.$	$M_3.$

SECOND METHOD. We may divide each member at once by the coefficient of n, namely $\frac{2}{3}$. Remember that dividing 8 by $\frac{2}{3}$ is the same as multiplying 8 by $\frac{3}{2}$, the reciprocal of $\frac{2}{3}$. Hence we may write the solution as follows:

(1) $\frac{2}{3}\,n = 8.$	Given.
(2) $n = 12.$	Dividing by $\frac{2}{3}$ (Ax. 4).

Check. $\frac{2}{3} \times 12 = 8,$

or $8 = 8.$

NOTE. *Remember that* $\frac{2}{3}\,n$ has the same meaning as $\frac{2\,n}{3}$. Verify this statement by substitution. Similarly, $\frac{4}{5}\,a = \frac{4\,a}{5}$, and $\frac{3}{100}\,p = \frac{3\,p}{100}$.

Exercises. *Equations Solved by Using Two Processes*

Numbers 1 to 10, oral

Observe that in all the equations which you have solved thus far, you used the reverse of the operations which appeared in the original equation.

Solve the following equations and check each solution:

1. $2\,n + 1 = 11.$
2. $3\,n + 3 = 15.$
3. $4\,n + 6 = 66.$
4. $6\,n + 15 = 45.$
5. $4\,d - 1 = 11.$
6. $7\,h - 2 = 26.$
7. $10\,l - 7 = 13.$
8. $8\,r - \frac{1}{2} = 15\frac{1}{2}.$
9. $\frac{d}{4} + 3 = 6.$
10. $\frac{n}{5} + 7 = 9.$
11. $\frac{1}{10}\,h + \frac{1}{3} = \frac{1}{2}.$
12. $\frac{1}{4}\,p + 4.1 = 8.1.$
13. $\frac{n}{2} - 3 = 5.$
14. $\frac{w}{3} - 6 = 0.$
15. $\frac{1}{4}\,b - \frac{1}{3} = 4\frac{2}{3}.$
16. $\frac{1}{5}\,c - 7.1 = 8.9.$
17. $\frac{2}{3}\,a = 6.$
18. $\frac{3}{5}\,b = 9.$
19. $\frac{4}{7}\,d = 8.$
20. $\frac{3}{100}\,c = 12.$
21. $\frac{5\,n}{7} = 10.$
22. $\frac{6\,n}{100} = 0.$
23. $\frac{5}{8}\,x = 15.$
24. $2\frac{1}{2}\,x = 5.$
25. $3\frac{1}{3}\,d = 20.$
26. $4\frac{1}{5}\,n = 42.$
27. $12\frac{1}{2}\,d = 100.$
28. $\frac{7}{8}\,n = 14.$
29. $3\frac{1}{7}\,d = 22.$
30. $.04\,x = 8.$

31. $.05\,c = 40$.
32. $.06\,p = 12$.
33. $.03\,x = 24.66$.
34. $\frac{2}{3}\,d + 4 = 12$.
35. $\frac{3}{5}\,n - 4 = 2$.
36. $\frac{7}{8}\,x + 4 = 11$.
37. $\frac{8}{9}\,w - 1 = 15$.
38. $4\frac{1}{3}\,h + 3 = 16$.
39. $.1\,d = 1$.
40. $.3\,n = 6$.
41. $4\,n + 80 = 90$.
42. $6\,w - 2.6 = 19$.
43. $9\,a + \frac{1}{2} = 20\frac{1}{2}$.
44. $8\,d - 3 = 41$.
45. $\frac{n}{6} + 900 = 1000$.
46. $\frac{1}{7}\,d - 300 = 1$.
47. $\frac{15}{16}\,x + 3 = 18$.
48. $\frac{9}{10}\,r - 1\frac{1}{2} = 7\frac{1}{2}$.
49. $4\frac{1}{5}\,r + 9 = 51$.
50. $5\frac{1}{6}x - 30.4 = 31.6$.

Timed Practice Test

Time, 5 minutes

1. $4\,n + 3 = 15$.
2. $6\,n + 2 = 20$.
3. $5\,n + 7 = 27$.
4. $7\,n + 5 = 40$.
5. $2\,n + 1 = 3$.
6. $10\,x + 7 = 57$.
7. $11\,x + 4 = 70$.
8. $20\,x + 9 = 69$.
9. $17\,n + 2 = 53$.
10. $100\,n + 500 = 700$.
11. $8\,x + 7 = 23$.
12. $9\,x + 5 = 50$.
13. $12\,x + 4 = 76$.
14. $7\,x + 3 = 73$.
15. $4\,n + 2 = 9$.
16. $3\,n + 7 = 10$.
17. $5\,n + 2.1 = 12.1$.
18. $2.1\,x + 6 = 12.3$.
19. $3\,x + 1 = 2$.
20. $4\,x + 3 = 5$.

Timed Practice Test

Time, 5 minutes

1. $5\,n - 9 = 1$.
2. $6\,w - 4 = 8$.
3. $7\,w - 10 = 4$.
4. $8\,n - 1 = 15$.
5. $5\,n - 2 = 8$.
6. $3\,n - 15 = 6$.
7. $2\,n - 7 = 13$.
8. $5\,x - 6 = 24$.
9. $6\,x - 1 = 5$.
10. $2\,x - 5 = 5$.
11. $3\,y - 6 = 0$.
12. $5\,y - 1 = 14$.
13. $7\,y - 4 = 3$.
14. $10\,y - 7 = 13$.
15. $12\,y - 7 = 53$.
16. $20\,x - 1 = 59$.
17. $2\,x - 5.1 = 7.9$.
18. $3\,x - 4.3 = 8$.
19. $4\,x - 3 = 7$.
20. $7\,x - 4 = 5$.

Timed Practice Test

Time, 5 minutes

1. $\frac{n}{3}+1=5.$
2. $\frac{n}{2}+5=7.$
3. $\frac{n}{5}+1=8.$
4. $\frac{n}{3}+4=6.$
5. $\frac{n}{4}-1=6.$
6. $\frac{x}{7}-2=3.$
7. $\frac{x}{9}-1=10.$
8. $\frac{n}{2}+7=9.$
9. $\frac{w}{6}+5=8.$
10. $\frac{d}{10}+100=150.$
11. $\frac{5}{6}n=10.$
12. $\frac{5w}{9}=100.$
13. $\frac{3}{2}x=12.$
14. $\frac{2}{3}x+1=5.$
15. $\frac{3}{5}x+2=8.$
16. $\frac{3}{4}x-1=5.$
17. $\frac{5}{8}x-2=3.$
18. $\frac{8w}{5}=16.$
19. $\frac{m}{45}+6\frac{1}{3}=9\frac{1}{3}.$
20. $.06p=120.$

50. Composite Types of Equations. The solution of a given equation may, of course, require the use of more than two steps. In such a case we may apply the fundamental axioms repeatedly until we obtain the required missing number.

Illustrative Example. Solve the equation $6a+3=4a+5.$

Solution.

(1)	$6a+3=4a+5.$	Given.
(2)	$6a=4a+2.$	S_3 (Ax. 2).
(3)	$2a=2.$	S_{4a} (Ax. 2).
(4)	$a=1.$	D_2 (Ax. 4).

Check. $6\times 1+3=4\times 1+5,$

or $6+3=4+5,$

or $9=9.$

Exercises. *Composite Types of Equations*

1. $14a+60-2a=72.$
2. $3w+15+2w=18+4w.$
3. $6n+4-n=n+20.$
4. $12+n=7n+6.$
5. $8+2h=7h+3.$
6. $4x+4+6x=9x+6.$

7. $2x + 20 + 33x = 40 + 15x$.

8. $12d + 4 - 3d = 39 + 2d$.

9. $7n = 0$.

10. $100 + n = 9n + 20$.

11. $3x - 6 = 0$.

12. $\frac{1}{2}h = 0$.

13. $8x - 7 + 2x = 3$.

14. $17r - 34 = 0$.

15. $10n + 8 + 6n + 6 + 2n + 2 = 6n + 40$.

16. $4x + 8 + 2x + 7 - x = 15 + x + 10 + 2x$.

17. $9n + 7 + 3n + 5 = 8n + 22 - 2n - 10$.

18. $8n + 3 = 10n - 4$.

19. $9n + 7 - 6n = 12n + 4$.

20. $14n + 8 - 7n = 21n - 2$.

21. $3a + 4 = a + 6$.

22. $4b + 9 = 2b + 15$.

23. $7w - 6 = 2w + 4$.

24. $8r - 7 = 5r + 14$.

25. $9l + 2 = 8 + 7l$.

26. $8w + 6 = 9 + 5w$.

27. $4w + 6 = 5w + 3$.

28. $9a + 10 = 7a + 16$.

29. $10b + 4 = 12b + 2$.

30. $9w - 4 = 6 + 4w$.

31. $8a + 6 + 5a + 3 = 6a + 23$.

32. $2.8a + 6 + 9.2a = 12$.

33. $8x + 4 + 7x = 9x + 10$.

34. $9d - 7 + 2d = 14 + 4d$.

35. $8l + 5 - 4l = 23 - 5l$.

36. $7\frac{1}{2}a - 6 - 3\frac{1}{2}a = 24 - 6a$.

37. $9n + 7 + 6n + 3 = 40 - 5n$.

38. $8x + 7 - 6x - 3 = 19 - 7x$.

39. $9x - 8 = 0$.

40. $7d + 4 - 3d = 4$.

Exercises. *Algebraic Representation*

1. One number is 4 more than another number. If n denotes the smaller number, what algebraic expression represents the sum of the two numbers?

2. One number is five times as large as another number. If n represents the smaller number, what algebraic expression represents their sum? their difference?

3. One number is 3 less than another number.

(1) If the smaller number is n, what is their sum?

(2) If the larger number is n, what is their sum?

4. Mary and Elizabeth played a game. Mary's score exceeded that of Elizabeth by 5. What was the combined score of the two girls? (Use s for Elizabeth's score.)

EXPLANATION. The statement $8 - 6 = 2$ may be read in several ways:

(1) The difference between 8 and 6 is 2.

(2) 8 exceeds 6 by 2.

(3) 8 is 2 larger than 6.

5. Write in short form:

(1) The difference between a and b is c.

(2) x exceeds y by r.

(3) x is d larger than y.

6. A man's profit, in dollars, was p. How do you represent

(1) a profit which is \$1000 larger?

(2) a profit which is \$2000 less?

(3) a profit which is five times as great?

(4) a profit which is half as great?

7. Paul's age is a. His brother is 5 years older. Express their ages 10 years from now.

8. A dozen eggs cost c cents. What is the cost of one egg? of 5 dozen eggs?

9. The cost of a suit, in dollars, was d. What is the price of 4 such suits? of n suits?

10. The cost of 4 bushels of potatoes was $3.60. What was the price of 1 bushel? of n bushels?

11. During a certain week William earned d dollars. He spent c dollars for lunches and 50 cents for his other expenses. How many cents did he have left?

12. I bought a house having a value of n dollars. The interest rate on this sum was 6% a year. My other expenses were $400 a year. How many dollars did the house cost me per month?

13. Remember that integers which are arranged in their natural order, such as 6, 7, 8, are called *consecutive* integers. If n represents any integer, how do you represent the next two consecutive integers?

14. Study the following table of *even* integers:

$$2 = 2 \times 1 \qquad 8 = 2 \times 4$$
$$4 = 2 \times 2 \qquad 10 = 2 \times 5$$
$$6 = 2 \times 3 \qquad 12 = 2 \times 6$$

Evidently, any even integer may be written in the form $2 \times (\ \ )$, or $2n$. Explain.

15. If $2n$ represents any even integer, what expression represents the next consecutive *even* integer?

16. Study the following table of *odd* integers:

$$7 = 6 + 1 \qquad 21 = 20 + 1$$
$$9 = 8 + 1 \qquad 25 = 24 + 1$$

Evidently, any odd integer may be represented by the expression $2n + 1$. Explain.

17. If $2n + 1$ represents any odd integer, write an expression for the next odd integer.

51. Summary. In this chapter you have learned

(1) How to solve equations by inspection.

(2) How to solve equations by rule.

(3) How to check the solution of an equation.

(4) The use of the fundamental axioms.

(5) The meaning of important terms, such as ***equality, identity, equation, root, solution, members of an equation.***

Test on Chapter II

"The Case of A and B"

In the following contest imagine that two boys are playing a game and that A represents the score of one boy, while B represents the score of the other boy. The class may be divided into two teams. The team which scores the largest number of correct answers wins the equation contest.

1. $A + B = 20$

If A is	1	3	7	10	12	19
then B is . . .	—	—	—	—	—	—

2. $A - B = 40$.

If B is	3	7	10	15	20	39
then A is . . .	—	—	—	—	—	—

3. $A + 2B = 30$.

If A is	2	6	8	10	14	28
then B is . . .	—	—	—	—	—	—

4. $4A - 2B = 20$.

If B is	0	2	4	8	12	40
then A is . . .	—	—	—	—	—	—

5. $\frac{4}{5}A + 4B = 60$.

If A is	0	5	10	35	25	50	
then B is . . .	—	—	—	—	—	—	

6. $\frac{3}{4}A + 6B = 72$.

If A is	0	8	16	40	32	48	56
then B is . . .	—	—	—	—	—	—	—

7. $\frac{3}{4}A - 3B = 9$.

If B is	1	2	7	5	8	12	20
then A is . . .	—	—	—	—	—	—	—

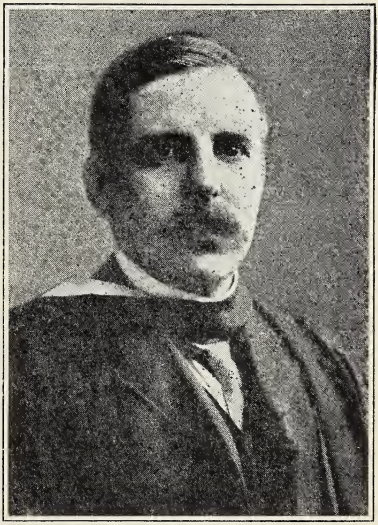
8. $\frac{2}{3}A - B = 21 + 2B$.

If B is	1	3	5	7	11	21
then A is . . .	—	—	—	—	—	—

CHAPTER III

PROBLEMS

52. Solving Problems by Equations. Many mathematical problems, as you have seen, can be solved very quickly by using equations. In fact, the equation is the principal tool for solving such problems. In shops, laboratories, offices, banks, and factories the use of the equation is a great timesaver. It is consequently very important that you should have considerable practice in the solution of problems by equations.



A picture of Lord Rutherford of Nelson, one of the foremost scientists of our time, who did important work in radioactivity while at McGill University

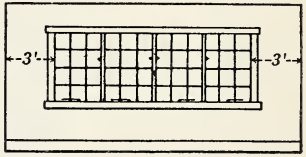
The problems that arise in everyday life are often very complicated or technical. Such problems require a knowledge of special facts or principles which cannot be taught in ordinary algebra classes. It will

be our aim, instead, to consider only problems which do not depend on technical knowledge and which can be mastered without long-continued special study. While these simpler problems are not always very "practical," you will find that the *method* used in solving them applies also in the case of more technical problems.

53. Simple Applied Problems. Some of the problems which follow can be solved by either arithmetic or algebra. You may work them out by arithmetic first, if you prefer. Study the illustrative examples and try to discover, in each case, in what way the equation method is better than the arithmetic solution.

Illustrative Example 1. Four casement windows of equal width are to be built in a wall 18 feet long, leaving a space of 3 feet on each side of the windows. What must be the width of each window?

Solution. 1. The missing number in this case is the *width* of each window. Hence we represent it by w.



2. We see that four of these widths, increased by 6 feet, make up the whole length of the room, which is 18 feet.

3. This relation may be written in equation form, as follows:

$$4w + 6 = 18.$$

4. Solving this equation, we have

(1)	$4w + 6 = 18.$	Given.
(2)	$4w = 12.$	S_6.
(3)	$w = 3.$	D_4.

5. We see that this solution is correct, for if we have four widths of 3 feet each, and add 6 feet, we obtain a total distance of 18 feet.

Try to summarize the five steps used in this solution.

Illustrative Example 2. John, Henry, and Fred together built a chicken house and started a small poultry business. Henry contributed as much money to the business as John, while Fred contributed as much as John and Henry together. At the end of the year the total profit of the business was found to be \$80. How much of this profit should each boy receive?

Solution. 1. Let John's profit, in dollars, be represented by p.

2. Then Henry's profit is also represented by p, while Fred's profit is represented by $2\,p$. (Why?)

3. The total profit is then represented by $4\,p$. Hence we have the equation $4\,p = 80$.

4. Solving this equation, we have

(1)	$4\,p = 80.$	Given.
(2)	$p = 20.$	D_4.

5. Since John's share is \$20, Henry's share is also \$20, while Fred's share is \$40. Also, since \$20 + \$20 + \$40 = \$80, the solution is correct.

The work for this problem may be given in condensed form, as follows:

Boy		Profit	Check	
John		p	= \$20	
Henry		p	= \$20	
Fred		$2\,p$	= \$40	
Solution.	1.	$4\,p$	= 80	Given.
	2.	p	= 20	D_4.

54. Steps Used in Solving Problems. The above illustrative examples suggest the following steps, which aid in the solution of problems stated in words:

1. Read the problem carefully. Picture the situation which it describes. Try to discover what is *required*. Represent the unknown quantity by a letter.

2. If several unknown quantities are to be found, express their *relations* in terms of a *single* letter. If convenient, make a *table* showing these relations.
3. Form the *equation* suggested by the conditions of the problem.
4. *Solve this equation* and thus find the results required.
5. *Check* your results by using the original statements of the problem.

KEY WORDS: **missing numbers, relations, equation, solution, check.**

NOTE. It is very important to distinguish carefully between checking the solution of an *equation* and checking the solution of a *problem*. In the case of a problem, it is necessary to go back to the *original* statements to see whether the solution "works." For it is possible that you may have solved the equation correctly, but the equation itself may not correctly represent the conditions mentioned in the problem. You would not discover this mistake by merely checking the solution of the equation.

Exercises. *Problems*

Solve each of the following problems, using the suggestions given above.

Draw a figure whenever it is convenient. For the sake of comparison, try to solve Exercises 1 to 9 by both arithmetic and algebra.

1. A rectangular field, 100 feet wide, is to be enclosed with wire fencing. If 500 feet of fencing are at hand, how long can the field be made?

2. The length of a rectangular field is to be three times the width. Suppose that 80 rods of fence are at hand to enclose it. What must the dimensions of the field be?

3. A box is 3 feet long and 2 feet wide. How high must it be to hold 12 cubic feet?

4. Mr. Jones has on hand 100 feet of poultry netting, which he wishes to use to make a chicken yard.

(1) What would be the dimensions of the yard if he made it square?

(2) What would the dimensions of the yard have to be if the length were to be three times the width?

(3) Which arrangement would make the larger yard?

5. In my garden I wish to have a border for flowers, which is to be 3 feet wide. I have estimated that the flowers and plants which I wish to grow during the coming season will require 60 square feet of garden space. How long must the border be?

6. How high should a schoolroom be which is 30 feet long and 20 feet wide if it is to accommodate 36 pupils, 200 cubic feet of air space being considered necessary for each pupil in a schoolroom?

7. Henry and his sister were together given 90 cents each day to pay their school expenses. They had to pay 16 cents each for carfare and with the remainder had to buy their lunch. How much can each spend for lunch?

8. The graduating class of the Eastwood High School wishes to buy a class memorial, which is to cost $120. If there is $50 in the class treasury, how much should each of the 140 members of the class contribute?

9. In a certain city the taxi rate is 30 cents for the first mile and 20 cents for each additional mile.

(1) How far did a passenger travel whose fare amounted to $1.30?

(2) How far can a passenger ride if he wishes to spend not more than 90 cents for a taxi fare?

SUGGESTION. Let n = the number of miles travelled at the 20-cent rate.

55. Number-Relation Problems. Number problems, while of no practical importance, will serve to give you additional skill in analyzing and solving verbal problems. Study the following illustrative examples.

Illustrative Examples. Suppose that someone had written the whole numbers 1, 2, 3, 4, and so on, up to 100, and that you were asked to select from them certain numbers, provided these were described in some definite way. An indefinite number of problems could then be given, many of them of a type that could be solved readily by the equation method.

1. Select two of these numbers such that one is three times the other and their sum is 60.

Solution. Let	n = the first number.
Then	$3n$ = the second number.
Then	$4n = 60$. (Why?)
Hence	$n = 15$.
Therefore	$3n = 45$.

Check. Evidently, since 45 is three times 15, and since $15 + 45 = 60$, the solution is correct.

2. Select three numbers such that the second is 2 greater than the first, and the third is equal to the sum of the first two, provided the sum of all three numbers is 36.

Solution. Let n = the first number.
Then $n + 2$ = the second number.
Then $2n + 2$ = the third number. (Why?)

(1)	$4n + 4 = 36$.	(Why?)
(2)	$4n = 32$.	S_4.
(3)	$n = 8$.	D_4.

Hence the required numbers are 8, 10, and 18.

Check. (Do these numbers meet all the conditions of the problem?)

Exercises. *Number Relations*

1. Three whole numbers are represented by n, $n+2$, and $n+3$. Their sum is 26. Find the numbers.

2. Four whole numbers are represented by n, $2n+1$, $3n+2$, and $4n+6$. Their sum is 99. Find the numbers.

3. Four whole numbers are represented by n, $4n+1$, $2n$, and $n+9$. The sum of the first two equals the sum of the last two. Find the numbers.

4. Six times a certain number is equal to the number increased by 10. Find the number.

5. The greater of two numbers is seven times the less, and their sum is 160. Find the numbers.

6. When a certain number is multiplied by 7, and the product is decreased by 4, the result is three times the number. What is the number?

7. The sum of two numbers is 90, and the greater is eight times the smaller. Find the numbers.

8. Two numbers differ by 56, and the greater is eight times the smaller. Find the numbers.

9. Two whole numbers are such that one is 10 greater than the other and their sum is 40. Find the numbers.

10. The sum of two numbers is 28. The greater exceeds the smaller by 2. Find the numbers.

11. The sum of four numbers is 279. The first two numbers are equal. The third is twice the sum of the first two. The fourth is three times the second. What are the numbers?

12. Find two consecutive integers whose sum is 59.

SUGGESTION. The first integer is n. The second is $n+1$.

13. Can the sum of two consecutive integers ever be an even number? Explain.

14. Verify the relations shown in the following table:

Number of Consecutive Integers	Algebraic Form	Sum
1	n	
2	$n+1$	$2n+1$
3	$n+2$	$3n+3$
4	$n+3$	$4n+6$
5	$n+4$	$5n+10$
6	$n+5$	$6n+15$

15. Find three consecutive integers whose sum is 63.

16. Find five consecutive integers whose sum is 110.

17. Find two consecutive even integers whose sum is 62.

Suggestion. Let $2n$ represent the first integer.

18. Find three consecutive even integers whose sum is 66.

19. Make a table showing the sum of two, three, four, five, consecutive even integers.

20. Find two consecutive odd integers whose sum is 44.

Suggestion. Let $2n+1$ represent the first integer. Then $2n+3$ represents the second integer.

21. The sum of three consecutive odd integers is 129. Find them.

22. Which of the following numbers might represent

(*a*) the sum of three consecutive integers?
(*b*) the sum of three consecutive even integers?
(*c*) the sum of three consecutive odd integers?

(1) 36. (2) 60. (3) 75. (4) 249. (5) 126.

56. Geometric Problems. Many problems concerning perimeters, areas, and volumes can be solved more easily by using the equation method. The following exercises will serve to increase your skill in understanding and solving such problems.

Exercises. *Geometric Problems*

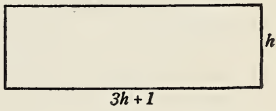
1. The perimeter of a square was found to be 100 feet. How long was one side? Solve by arithmetic and by algebra.

2. The perimeter of an equilateral triangle was equal to that of a square of side 3. Find the side of the triangle. Solve by arithmetic and by algebra.

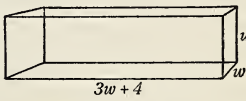
3. The sides of a triangle are represented by consecutive integers. Write a formula for the perimeter.

4. In Exercise 3, if the perimeter is 63, how long is each side?

5. A rectangle whose dimensions have the relations shown in the figure is such that if each dimension is increased by 5 the perimeter becomes 62. Find the dimensions of the rectangle.



6. It is required to make a wire model of a rectangular solid, the dimensions having the relations shown in the figure. If the total length of the wire available for this purpose is 40 inches, what must be the dimensions of the model?



7. An automobile driveway is 54 feet long and 8 feet wide. It is to be covered with cinders. If three loads of

cinders are available for the purpose, how deep a layer of cinders is possible? (1 load = 2 cubic yards.)

8. The base of a cylindrical can has an area of 14 square inches. How high must it be to have a volume of 70 cubic inches?

9. A group of boys decide to lay out a circular running track which is to be a quarter of a mile in length. What must be the radius of the track? ($C = 2\pi r$.)

10. The floor area of a rectangular store was 3200 square feet. If the store had a frontage of 120 feet, what was its depth?

11. The side of a square is $2s + 5$. If the perimeter is 100, find s and $2s + 5$.

57. Business Problems. Many business problems can be solved very easily by the use of equations. They are concerned with questions of cost, of profit and loss, of interest, discount, commission, and the like. Since percentage is of very common occurrence in such problems, we shall consider first the solution of equations having decimal coefficients.

Illustrative Example. Find a number of which 15 is 3 per cent.

Solution. Let n = the required number.

Then $3\%\, n = 15$.

This may be written $.03\, n = 15$,

or $\frac{3}{100}\, n = 15$.

Dividing both sides by $\frac{3}{100}$, we have

$$n = 500.$$

Check. $\frac{3}{100} \times 500 = 15$,

or $15 = 15$.

Exercises. *Business Problems*

1. $5\% n = 20$.
2. $6\% n = 24$.
3. $4\% x = 16$.
4. $4\% m = 60$.
5. $10\% x = 90$.
6. $12\% x = 84$.
7. $2\% n = 4$.
8. $3\% n = 9$.
9. $2\frac{1}{2}\% n = 10$.

SUGGESTION. Remember that $2\frac{1}{2}\% n = \frac{2\frac{1}{2}}{100} n = \frac{5}{200} n$.

10. $3\frac{1}{2}\% x = 21$.
11. $3\frac{1}{3}\% n = 10$.
12. $4\frac{1}{4}\% c = 17$.
13. $2\frac{1}{8}\% x = 34$.
14. $3\frac{3}{4}\% x = 30$.
15. $x + .04\, x = 208$.

Solution. Exercise 15 may be written $1.04\, x = 208$,

or $\frac{1\,0\,4}{1\,0\,0} x = 208$.

Dividing by $\frac{1\,0\,4}{1\,0\,0}$, we have $x = 200$.

16. $x + .03\, x = 309$.
17. $n + .06\, n = 106$.
18. $c + 4\% c = 312$.
19. $x + 15\% x = 805$.
20. $p + 12\% p = 336$.
21. $x - .04\, x = 96$.

SUGGESTION. Exercise 21 may be written $.96\, x = 96$.

22. $n - 10\% n = 270$.
23. $x - 2\% x = 196$.
24. $c - 8\% c = 368$.
25. $x - .06\, x = 188$.

26. In what time will a sum of \$4000 yield an income of \$800, at 4% a year?

SUGGESTION. Use the formula $i = prt$.

Hence $4000 \times .04 \times t = 800$.

27. In what time will a sum of \$2500 yield an income of \$375, at 5% a year?

28. At what rate must \$4000 be invested in order to yield a profit of \$200 a year?

SUGGESTION. $\frac{r}{100} \times 4000 = 200$.

29. At what rate must \$3500 be invested in order to yield \$210 a year?

30. Using the formula $i = prt$, find, by the equation method, the missing values in the following table:

	Principal	Rate	Time	Interest
(1)	\$500	4%	2 years	
(2)	\$600	4%		\$72
(3)	\$940		2 years	\$94
(4)		6%	$1\frac{1}{2}$ years	\$97.20

Using the percentage formula $p = br$, find the missing values in the following exercises:

31. $b = 200$, $r = 5\%$.
32. $b = 88$, $r = 10\%$.
33. $b = 916$, $r = 12\%$.
34. $p = 18$, $r = 5\%$.
35. $p = 40$, $r = 2\%$.
36. $p = .84$, $r = 6\%$.
37. $p = 8$, $b = 40$.
38. $b = 2.50$, $p = 5$.
39. $b = 9.16$, $p = 36.64$.
40. $b = 8140$, $r = 2\frac{1}{2}\%$.

41. In a class of 40 pupils 9 failed. What per cent of the class failed?

42. On a certain farm the gross receipts during a certain year were \$10,000, while the total expenses were \$6270. What per cent of the receipts were the expenses?

43. A man bought a house for \$5500 and sold it for \$6200. What was his per cent of gain?

44. A farmer took 940 lb. of milk to a creamery. It was found that the milk produced 38.5 lb. of butter fat. What was the per cent of butter fat?

45. A pair of shoes sold for \$4.50, after a discount of 10% had been allowed on the marked price. What was the marked price?

46. At what price must a pair of shoes costing \$6 be listed in order that after allowing a discount of 10% the firm may yet make a profit of 20%?

Solution. Let l denote the list price.

Then $.90\,l = 1.20 \times 6.$

This may be written $\frac{90}{100}\,l = \frac{120}{100} \times 6.$

Dividing by $\frac{90}{100}$, we have $l = 8.$

That is, the list price is \$8.

Check. 90% of \$8 = \$7.20.

120% of \$6 = \$7.20.

47. A man bought an automobile for \$960. The following year he sold it for \$500. What was his per cent of loss?

48. A house sold for \$5300, after a discount of 4% had been allowed for cash. What was the original price?

49. At what price must a contractor offer a house that cost him \$5700, in order that after allowing a selling agency a commission of 5% he may make 10% himself?

50. An automobile agent receives a commission of 25%. What must be the list price of the cars he sells if his firm is to make a profit of 20%, the cost of a car being \$600?

58. Everyday Problems. The problems of life are never accompanied by a rule that tells how they should be solved. Instead, one must *discover* in each case exactly what is required and what must be done to solve the problem.

The following problems, while not practical in the usual sense, are intended to give you some skill in understanding or picturing *described* situations, and in learning to apply to them the equation method.

Exercises. *Miscellaneous Problems*

1. A boy started on a trip from Edmonton to Evansburg, a distance of 72 miles. He rode eight times as far as he walked. How far did he walk?

2. Some Boy Scouts took a hike of 24 miles. They walked 3 miles farther in the morning than in the afternoon. How far did they walk in the afternoon?

3. A girl paid \$5.50 for a tennis racket and three balls. The price of the racket was \$4.75. Find the price of a ball.

4. The principal of a school, his wife, and his little daughter went to Vancouver. The total railroad fare amounted to \$42.20. If the child paid half fare, what was the price of each ticket?

5. Two boys having a launch took a party on a pleasure trip for \$25. The boy who owned the launch received \$5 more than the other boy. How much did each receive?

6. Three grades raised \$160 for the Scholarship Fund. The eighth grade raised \$10 more than the seventh grade, and the ninth grade raised \$20 more than the eighth grade. How much did each grade raise?

7. Three girls in the Commercial Department agreed to rent a typewriter for \$12. The second girl used it three times as many hours as the first, and the third girl twice as many hours as the first. How much should each pay?

8. Three basket-ball teams played 18 games. The first two teams won an equal number of games, and the third team won 4 games. How many games did each team win?

9. A baseball nine has played 12 games and has won 3 less than twice the number lost. How many games has it won?

10. Harry has \$18.75. He earned \$3 of this money by washing a car and the rest of the money by mowing lawns. He received 75¢ for each lawn which he mowed. How many lawns did he mow?

11. Three boys sold 70 tickets for a football game. John sold twice as many as Bob, and Dick sold four times as many as Bob. How many did each boy sell?

12. In an athletic meet three boys represented East High School. Together they scored 22 points. The first boy scored 5 more points than the second, and the third scored as many as the other two together. How many points did each one score?

13. I bought some 5-cent stamps and three times as many 2-cent stamps. If I paid 55¢ for all, how many of each did I buy?

14. How can a bill of \$3 be paid with half-dollars and quarters by using the same number of each?

SUGGESTION. Since the value of one half-dollar is 50 cents, the value of n half-dollars is 50 n cents.

15. From Toronto to Edmonton is 1980 miles. Joe travelled three times as far by boat as by bus and seven times as far by rail as by bus. How far did he travel by each method?

16. Three neighbors compare their taxes. The first finds that he pays twice as much as the second, and the third finds that he pays as much as the other two together. If the sum of their taxes is \$1440, how much does each pay?

17. Four boys decided to run a refreshment booth one summer. They found that the equipment would cost \$85. The father of one of the boys gave them \$25. How much will each boy have to pay if the expense is divided equally?

18. A man finds that it will cost him \$450 to build a garage. The materials cost \$290. A carpenter and his helper agree to do the work in 10 days. The carpenter is to be paid three times as much as his helper. How much does each earn per day?

19. Two boys went fishing and caught 15 fish. When they returned home, the boy coming from the larger family took 3 more than twice as many as the other one. How many did each boy take?

59. Some Old-Fashioned Problems. The problems which follow appeared in several algebra textbooks that were in use about seventy-five years ago. You will observe that some of them are very simple, while others may at first seem hard to you.

Exercises. *Old-Fashioned Problems*

1. John and Charles have 12 apples between them, and each has as many as the other. How many has each?

2. James and John have 24 peaches, and one has as many as the other. How many has each?

3. In a fruit basket there are three times as many apples as pears, and five times as many peaches as apples, — in all, 95. How many of each sort are there?

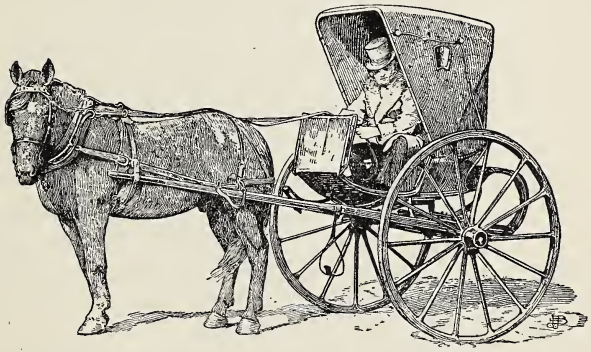
4. On a certain day a merchant paid out \$2500 to three men, A, B, and C; he paid to A a certain sum, to B \$500 less than the sum paid to A, and to C \$900 more than he paid to A. Find the sum paid to each.

5. A boy bought an equal number of apples, oranges, and pears for 96 cents; he bought the apples at 3 cents apiece, the oranges at 4 cents, and the pears at 5 cents. How many of each kind did he buy? *Ans.* 8.

6. A man bought 3 oxen, 4 cows, and 6 calves for 260 dollars. He paid twice as much for an ox as he did for a cow, and twice as much for a cow as for a calf. How much did he give for each?

Ans. For a calf, $10; for a cow, $20; and for an ox, $40.

7. A gentleman purchased a horse, a chaise, and a harness for $230. The harness cost a certain sum, the



chaise three times as much as the harness, and the horse $20 more than the chaise. Can you find the price of each?

8. A man paid a debt of $29, in three different payments; the second time he paid $3 more than he did the first time, and the third time twice as much as the second time. What was the amount of his first payment? *Ans.* $5.

9. A farmer hired a man and two boys to do a piece of work. To the man he paid $12, to one boy $6, and to the other boy $4 per week. They all worked the same length of time and received $264. How many weeks did they work?

10. One half of a post stands in the mud, one third in the water, and the remainder, which is 3 feet, is above the water. What is the whole length of the post?

(Let $6x$ = the length of the post.) *Ans.* 18 feet.

11. A farmer has three pastures for sheep. In the second pasture he has twice as many sheep as in the first, and in the third as many as in the first and second less 15, and he has in all 57 sheep. How many sheep has he in each pasture?

12. Jane has a blush rosebush, a moss rosebush, and a white rosebush, and there are in all 33 buds. The buds on the second rosebush are double those on the first, and those on the third four times those on the second. How many buds are there on each rosebush?

13. Four boys, A, B, C, and D, in counting their money, found they had together \$1.98, and that B had twice as much as A, C as much as A and B together, and D as much as B and C together. How much money did each boy have?

Ans. A, 18 cents; B, 36 cents; C, 54 cents; and D, 90 cents.

14. A man deposited in a savings bank, in three deposits, a total of \$80. The second deposit was double the first, and the third was equal to the first and second and \$8 over. What was the sum deposited each time?

15. A man had six sons, to whom he gave \$120, giving to each one \$4 more than to his next younger brother. How many dollars did he give to the youngest? *Ans.* \$10.

16. A person goes to an inn, where he spends 3 shillings. He then goes to a second inn and spends 9 shillings, which is three times as much as he had left. What had he at first?

17. James has twice as many dime pieces as cent pieces, and 42 cents in all. How many dime pieces has he?

18. Divide \$1000 among A, B, and C so that A shall have \$72 more than B, and C \$100 more than A.

60. Summary. In this chapter you have learned

(1) How to solve simple number problems.
(2) How to solve simple geometric problems.
(3) How to solve business problems.
(4) How to solve other applied problems.

Supplementary Honor Work

1. Make a collection of number-puzzle problems.

2. Make a collection of problems found in old algebra textbooks.

3. Try to make a list of original problems suggested by those given in this chapter.

Test on Chapter III

In each of the following problems write the algebraic expression which represents the answer:

1. James and Henry together have s dollars. James has \$16. How many dollars has Henry?

2. In how many hours can a boy walk d miles at the rate of r miles an hour?

3. Represent each of three consecutive numbers, and also their sum, if the first number is represented by n.

4. On a certain test Jane's and Harry's standings differed by $d\%$. If Jane received the lower rating, which was 75%, what was Harry's standing?

Under each of the following problems three equations are given. Select the equation which correctly represents the conditions of the given problem.

5. In a class of 32 pupils the number of girls was 6 less than the number of boys. How many boys and how many girls were there in the class?

(1) $n - 6 = 32$. (2) $2n - 6 = 32$. (3) $n + 6 = 32$.

6. A certain rectangular room required 72 feet of moulding. The room was twice as long as it was wide. Find its dimensions.

(1) $2w = 72$. (2) $4w = 72$. (3) $6w = 72$.

7. John has \$3.00 in the school bank at the present time, and he saves 25 cents a week. In how many weeks will he have \$10.00?

(1) $25w + 3 = 10$. (2) $25w + 300 = 1000$.
(3) $25w + 3.00 = 10.00$.

8. The side of an equilateral triangle is 3 inches longer than the side of a square. The two perimeters are equal. Find the side of each figure.

(1) $3s + 9 = 4s$. (2) $4s + 3 = 3s$. (3) $3s = 4s + 12$.

Write the equation which represents the conditions of each of the following problems:

9. One man has three times as much money as another. Together they have \$5400. How much money has each?

10. A farm of 325 acres is to be divided between two brothers. The older is to receive 75 acres more than the younger. How many acres should each receive?

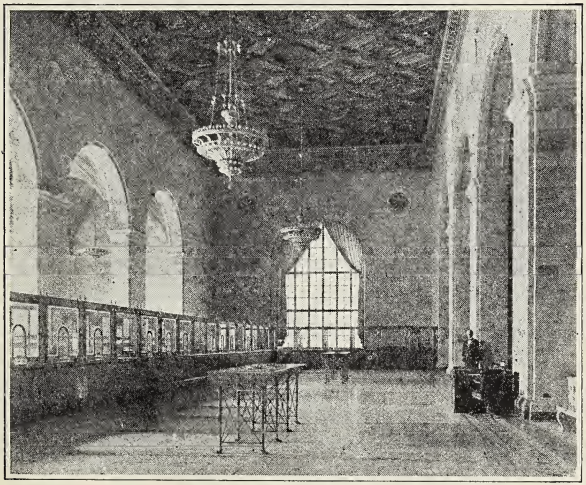
11. After a discount of 10% on the list price had been allowed, the cost of a radio set was \$126. What was the list price?

Solve completely each of the following problems:

12. I have 200 feet of wire fencing with which I wish to enclose a rectangular garden plot. This plot is to be 40 feet wide. How long may I make the plot if all the wire is to be used?

13. A class of 36 pupils, having only \$10 in its treasury, desires to purchase for the school a memorial valued at \$19. How much must each pupil in the class contribute so that the desired gift can be purchased?

14. I bought some 2-cent stamps and five times as many 1-cent stamps. If I paid 70 cents for all, how many of each did I buy?



Interior of head office of Royal Bank of Canada, Montreal

Courtesy of Royal Bank of Canada

CHAPTER IV

GRAPHS

1. Statistical Graphs

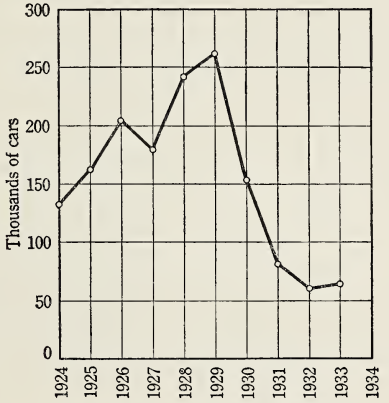
61. Importance of Statistics. Modern life has become so complicated that unless a record were kept of important facts, they would soon be forgotten, and great confusion would result. Thus, in school the attendance is taken in every class from day to day. The marks of each pupil are entered in the class books of the teachers and in the general office files. A business man, in order to keep track of the progress of his business, must constantly watch the amount of sales, expenses, net profits, and the like. Income-tax returns must be based on carefully kept accounts. The Census Bureau records the growth of the population and many other items. A large number of clerks are necessary in many offices, both private and public, to take care of the nation's bookkeeping.

For these reasons it is sometimes said that we are living in a statistical age, and it is not surprising that the importance of accurate statistics is being realized more clearly from year to year.

62. Why Graphs are Used. The statistical material which appears in newspapers and magazines is usually presented in *tabular* form. The *table* is undoubtedly the most convenient form of arranging statistical facts.

Nevertheless, when a statistical table is long, or contains large numbers, it is not always easy to understand the

story which the table contains. It has therefore become customary to "picture" such a table by making a drawing that represents the numbers found in the table. A drawing of this kind is called a *graph*.



Annual production of cars and trucks in Canada

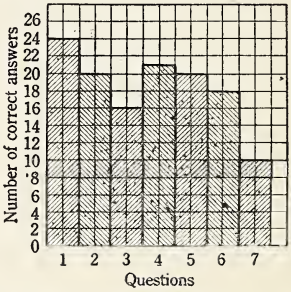
1924	132,580	1929	262,625
1925	161,970	1930	153,372
1926	204,727	1931	82,559
1927	179,054	1932	60,789
1928	242,054	1933	65,582

Study the graph shown above and the table which accompanies it. Which tells the story of the automobile industry more quickly?

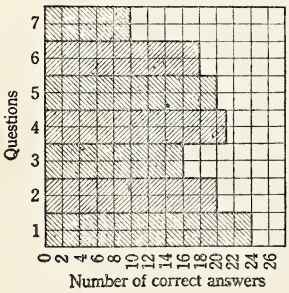
63. Kinds of Graphs. You may have observed that many other kinds of graphs are used in business and everyday life. Each graph tells a story and is based on some definite facts. The form which is chosen in any particular case depends on the facts to be pictured and on the purpose of the graph. Thus, if a graph is to be used for *advertising*, it

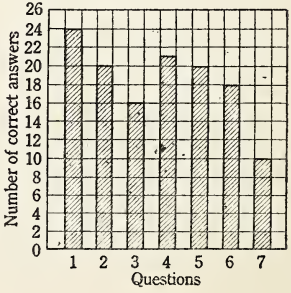
is necessary to keep in mind the distance at which it is to be clearly seen. A graph appearing in a newspaper is not necessarily effective when used on an advertising card in a street car. Begin to make a collection of graphs which you may find in newspapers, magazines, and advertising folders. Try to understand in each case the exact meaning of the graph and the method employed in making it. A few of the most important kinds of graphs will be considered in the following pages.

64. Bar Graphs. The diagram shows one form of *bar* graph. It is based on the following "story." A group of 25 pupils took a test of seven questions. The results were to be pictured graphically so that all could see at a glance how many pupils had answered each question correctly. Can you read, or interpret, the graph?



1. How many pupils answered the fifth question correctly?





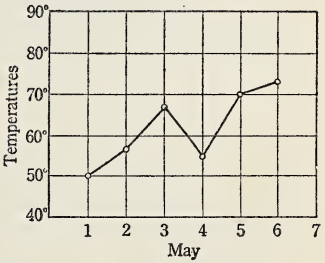
2. Which question caused the greatest difficulty?

3. Which question received the greatest number of correct answers?

In a bar diagram the bars may be arranged horizontally or vertically. They may be placed in a compact form or they may be separated. Which of the three forms shown on the preceding page would be the most effective one when seen at a distance?

65. Broken-Line Graphs. During the first six days of May, in a certain city, the average daily temperatures were as follows:

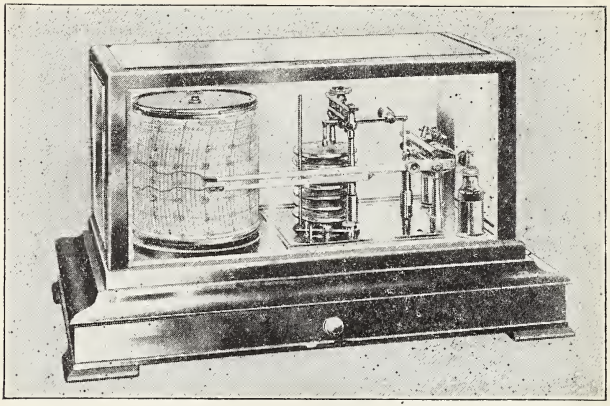
May 1	50°
May 2	57°
May 3	67°
May 4	56°
May 5	70°
May 6	73°



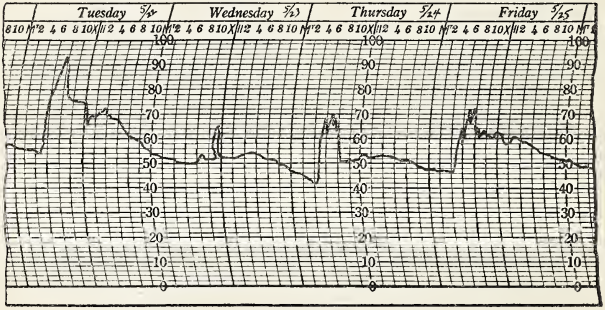
The adjoining graph pictures this table.

The line on which the days are marked off is called the *horizontal axis*, while the *vertical axis* shows the temperatures. The average temperature for each day is indicated by a point. By connecting these points in succession, as shown, we obtain a *broken-line* graph.

66. Curved-Line Graphs. When the facts to be pictured are very numerous, or when they fluctuate very much, it is often preferable to use a curved line in making the graph. Such a curve may be drawn free-hand or by means of special instruments. Thus, the temperature prevailing out of doors may change considerably from hour to hour. A complete record of these changing temperatures is made



The instrument shown in the picture makes a continuous graphic record of the temperature and of the barometric pressure



A temperature graph made by a thermograph

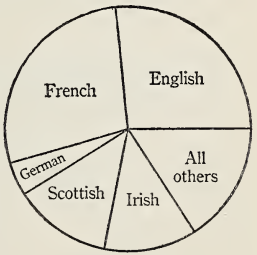
by a *thermograph*, which traces graphs, such as those shown above, on a special kind of graph paper.

One advantage of a *continuous* temperature record presented by a curved-line graph is that one may obtain from

it the temperature that prevailed at any time during the period which is considered. In a broken-line graph one cannot be equally certain about values that lie *between* those which were used in constructing the graph.

67. The Circle Graph. When it is desired to show the relation of each item of a table to the *sum* of the items, the graphic representation which is commonly used is the *circle* graph. The adjoining diagram illustrates a common form of the *circle* graph.

It is based on the following story. In Canada we have peoples of many racial origins. There are, for example, 28.2 per cent of French origin, 26.4 per cent of English origin, 13 per cent of Scottish origin, 11.9 per cent of Irish origin, and so on.



Observe that the circle graph shows these facts at a glance, giving clearly the *relative* importance of each item. It is for this reason that circle graphs are frequently used in business to show each item of the total cost of production, the relative wages paid to different classes of workmen, the effect of various types of advertising, and the like.

68. Interpretation of Graphs. By means of a graph it is possible to present numerical facts quickly and in a striking way. That is why graphs are used so extensively for commercial purposes. But just as we should always use common sense in reading a newspaper article or a book that may be of interest to us, so it is necessary to guard constantly against misleading or incorrect impressions in reading a

graph. This means that it is necessary, while reading a graph, to ask questions such as the following:

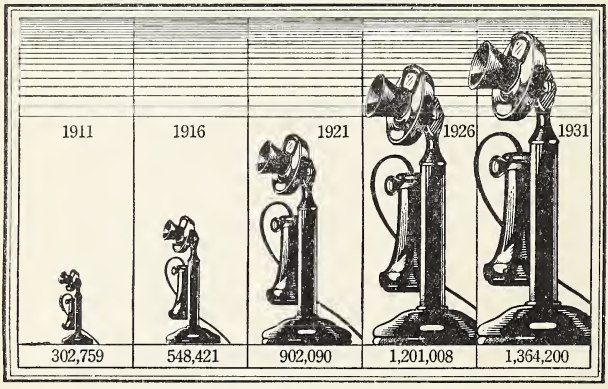
1. Does the story of the graph seem reasonable?

2. Is the graph constructed correctly?

3. Which parts of the graph have a message of particular interest?

Exercises. *Reading and Interpreting Graphs*

Examine each of the following graphs. Select any three of them for special study. Try to learn something about the story which each is supposed to tell, and then answer the accompanying questions.



Growth of the telephone business

1. The Canadian telephone companies expect eventually to "enable anyone anywhere at any time to pick up a telephone and talk to anyone else, quickly and at a reasonable cost."

(1) How does the graphic picture on the preceding page show the increase in the number of telephones used in this country from 1911 to 1931?

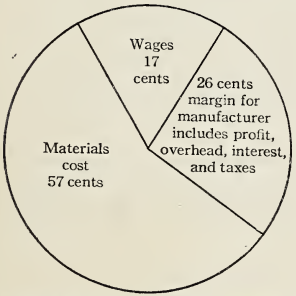
(2) Is a graphic picture of this kind as accurate as a bar diagram?

(3) Mention one advantage and one disadvantage of a pictorial graph. Why are such graphs used especially in advertising?

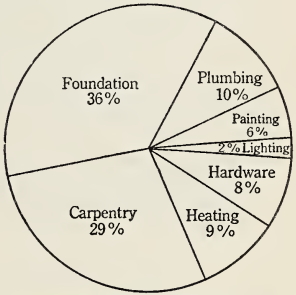
(4) What was the growth in the number of subscribers during each five-year period indicated in this graphic picture?

(5) Assuming the same rate of increase as during the last five-year period, estimate the number of telephones in use in Canada in 1936; in 1941.

2. Study the two adjoining circle graphs.



Manufacturer's dollar. Average division of industrial receipts

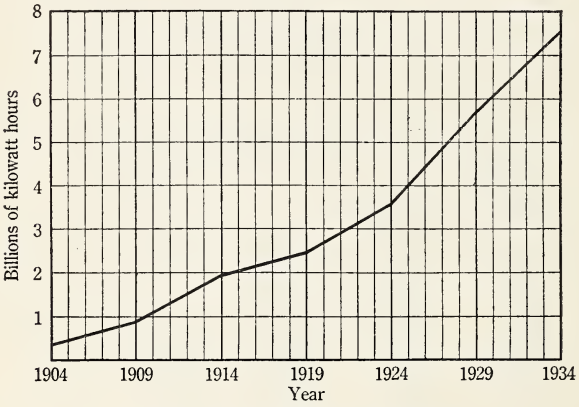


Building costs. Principal items of expense in construction

(1) Out of every dollar received by the manufacturer, according to the circle graph shown above, how many cents were expended for materials? How many for wages? How many cents remained for the payment of overhead expenses, taxes, and the like?

(2) Suppose that the total expense of building a new home is estimated at $10,000. According to the circle graph shown on page 101, what part of this sum must be set aside for the foundation? What part for carpentry? What part for plumbing?

3. Business men constantly study graphic charts such as the one shown below. Such charts are often called the "barometers of business." Why?



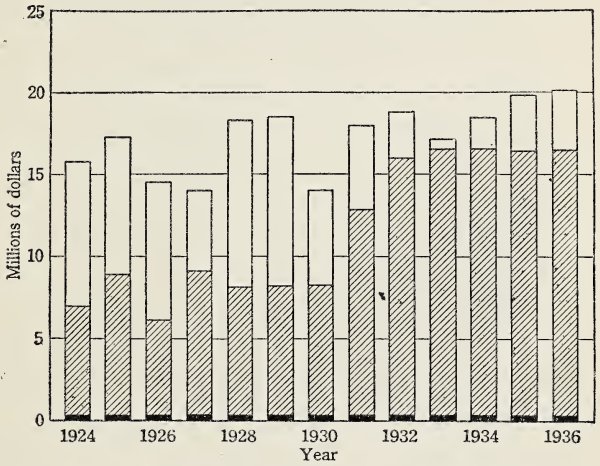
Electrical turbine horse power installed

(1) Statistics show that since 1904 the electrical turbine horse power installed has increased to about twenty-one times the original amount. How is this fact shown in the graph?

(2) Can you give a reason for this remarkable increase?

(3) Which five-year period showed the greatest increase in the installation of turbine electric power?

4. The bar diagram below shows the annual earnings of a large corporation since 1924, and the portion of this income which was paid out in the form of dividends.



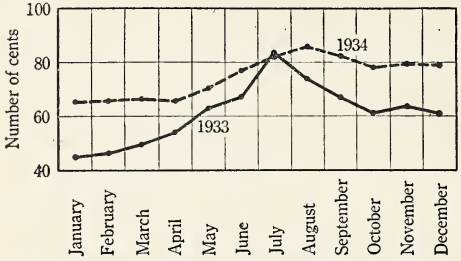
Annual earnings of a large corporation. In this chart the full height of each bar represents the total earnings in that year. The black, or lowest, section shows the amount paid out for preferred dividends, while the shaded section shows the amount of common dividends

(1) Which year showed the most marked increase in the payment of dividends?

(2) What is the outlook for continued prosperity in the case of this corporation?

5. The price of wheat is a very important factor in the life of Canadians and affects business all over the Dominion. Compare the two broken-line graphs on the chart shown on the next page.

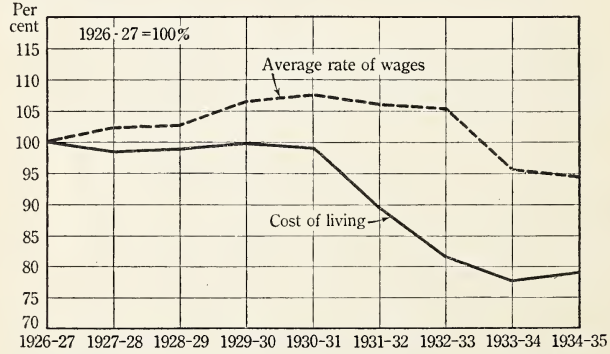
(1) How did the business of 1934 compare with that of 1933?



Monthly average prices per bushel at Winnipeg of No. 1 Northern Wheat — basis in store at Fort William and Port Arthur

(2) Find the highest and the lowest monthly averages.

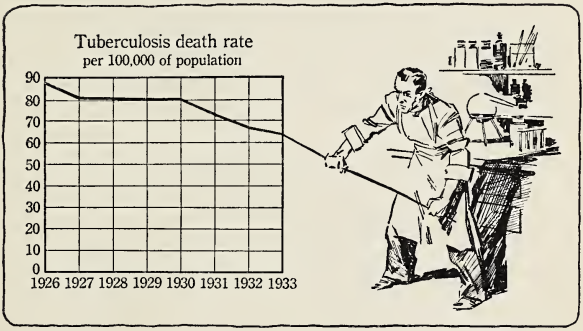
6. Study the comparative graphs shown on the following chart:



Cost of living and average rate of wages in Canada. Computed from data of Dominion Bureau of Statistics

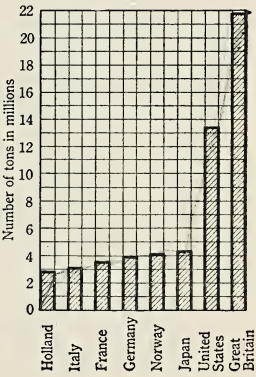
(1) What is true of the cost of living from 1929 to 1933?
(2) In what year was the cost of living lowest?
(3) In what year were wages highest?

7. Study the graphic chart shown below.



(1) Would this be a good poster for a health organization to use? Why? (2) How much did the death rate due to tuberculosis decline between 1926 and 1933?

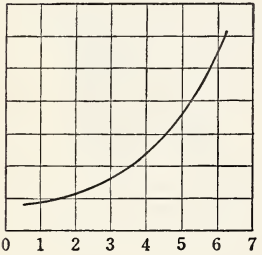
8. The graph at the right shows how the principal merchant fleets of the world compared in size in a recent year. By how many million tons did the merchant fleet of Great Britain exceed that of the United States? of Japan? Compare the tonnage of the United States with that of Japan; of Germany; of France.



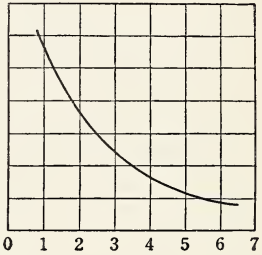
9. Examine each of the following typical graphs. Discuss the possible meaning of each if it is supposed to picture statistics relating to

(1) Population
(2) School attendance
(3) Income
(4) Expenses
(5) Cost of living
(6) Volume of business
(7) Health items
(8) Taxes
(9) Crops
(10) Exports

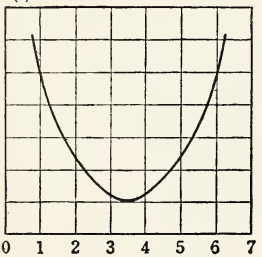
(1)



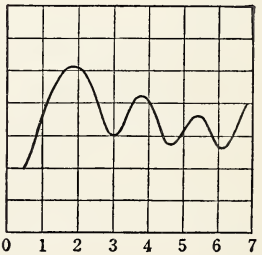
(2)



(3)



(4)



Thus, in the case of the cost of living, a graph such as (1) shows that prices are *going up*, while a graph of type (2) shows that prices are *going down*. A graph such as (3), however, means that prices decline and then rise again, while type (4) pictures *fluctuating* prices.

69. How to Make a Graph on Squared Paper. Graphs can usually be drawn most quickly and accurately by using *squared paper*. For this purpose it is possible to obtain squared paper ruled in inches or in centimetres. When the statistics to be pictured involve decimals, it is desirable to use decimally divided "inch paper" or "centimetre paper."

In most cases the graph is used to picture the facts given in a table, as was shown on page 95. In making a graph you will therefore find it helpful to keep in mind the following

Rules for Making Graphs

1. On a sheet of squared paper draw the axes. The *horizontal axis* is usually drawn at the bottom, and the *vertical axis* is drawn at the left side of the paper.
2. Using a convenient scale, mark off one set of numbers given in the table on the horizontal axis, and the other set on the vertical axis.
3. The divisions on each axis should be chosen in such a way that the numbers given in the table can be easily located.
4. It is usually best to start with zero, the numbers extending from zero to the right, and from zero upward, on the two axes.
5. In the case of a bar diagram, especially if it is to be used for advertising purposes, it is customary to draw the bars some distance apart. The space between the bars is usually made the same as the width of the bars.
6. The "story of the graph" should be explained by a suitable title written at the top or at the bottom of the graph sheet.

Exercises. *Making Statistical Graphs*

In the following exercises use any form of graphic representation which is found most convenient:

1. An observer at one of the weather-bureau stations reported the following hourly temperature readings for a day in July, from 6 A.M. to 6 P.M.:

Time	6	7	8	9	10	11	12	1	2	3	4	5	6
Readings . .	67°	69°	71°	77°	79°	82°	85°	86°	73°	80°	83°	82°	82°

Draw a graph picturing these temperature readings. On the afternoon of that day there was a thunderstorm. How does the graph show the resulting effect on the temperature?

2. Draw a temperature graph based on the figures given in the following table:

Day in September . .	1	2	3	4	5	6	7	8	9	10	11	12
Average temperature.	66°	57°	42°	40°	38°	32°	32°	31°	36°	43°	62°	50°

3. The average monthly temperatures reported during a recent year, from January to December, in one large city, were as follows: 27°, 24°, 28°, 40°, 54°, 62°, 70°, 70°, 61°, 49°, 40°, 25°.

Represent these figures graphically.

4. The average monthly rainfall in inches, from January to December, in another city, was reported as follows: 2.9, 2.6, 2.9, 2.5, 2.8, 3.1, 3.0, 2.9, 2.5, 2.8, 2.5, 2.7.

Draw a graph based on these figures. (Let 10 divisions of the vertical axis represent 1 inch of rainfall.)

5. The average heights of boys at various ages are shown in the following table:

Age in years . .	6	7	8	9	10	11	12	13	14	15	16
Height in inches	43.8	45.7	47.8	49.7	51.7	53.3	55.1	57.2	59.9	62.3	65.0

Draw a graph based on this table.

6. Represent graphically the statistical facts given in the following table, which gives the average heights of girls at various ages:

Age in years . .	6	7	8	9	10	11	12	13	14	15	16
Height in inches .	43.4	45.5	47.6	49.4	51.3	53.4	55.9	58.2	59.9	61.1	61.6

7. On a single sheet of squared paper, and using the same axes, draw the graphs of Exercises 5 and 6.

(1) During which years are the heights of boys and girls practically alike?

(2) During which years does the height of boys exceed that of girls?

8. The following table shows the average weights of boys and girls at various heights:

Height in inches	48	50	54	56	58	60	62	66	68	70
Weight of boys (in pounds) .	53	58	71	78	85	94	104	129	139	147
Weight of girls (in pounds) .	52	58	71	79	89	101	114	129	138	144

(1) Using the same set of axes, draw two comparative graphs, one representing the weights of boys, and the other the weights of girls.

(2) From your graphs determine the approximate weight of a boy 59 inches in height, and that of a girl 55 inches in height.

9. Eight algebra classes took a test on equations. Two weeks later they took the same test a second time. The table shows the average percentage of correctness obtained in these two trials.

Class	1	2	3	4	5	6	7	8
First trial	92	91	88	85	84	83	80	74
Second trial	97	94	90	86	84	82	78	70

(1) On a single sheet of squared paper, and using the same axes, draw comparative graphs which are based on these figures.

(2) How many of the classes improved their average on the second trial?

10. The following table shows the amount required in various years to purchase what one dollar bought in 1899:

1899	$1.00	1919	$2.57
1904	1.07	1924	1.91
1909	1.12	1929	1.33
1914	1.26	1934	1.37

Draw a graph based on this table.

11. The average retail price of certain foods is shown in the following table. The figures represent cents per pound.

Year	1926	1927	1928	1929	1930	1931	1932	1933	1934
Butter	41	42	42	43	37	27	22	22	24
Sirloin steak	29	30	35	36	36	29	25	21	21
Breakfast bacon	43	39	38	39	40	30	18	20	30

12. How the Canadian producer's average price of wheat has varied is shown in the following table:

1926	$1.09	1929	$0.88	1932	$0.29
1927	.84	1930	.41	1933	.41
1928	.67	1931	.32	1934	.50

Draw a graph representing these facts.

13. A census is taken in Canada every ten years. In millions the population of Canada, beginning with 1881 and ending with 1931, was as follows: 4.3, 4.8, 5.4, 7.2, 8.8, 10.3. Represent these statistics by a graph.

14. The amount of mail carried in airplanes in Canada for the years from 1929 to 1934, expressed in thousands of pounds, was as follows: 431, 474, 470, 414, 539, 625. Make a graph based on these figures.

SUGGESTION. When a statistical table contains large numbers, it is evidently impossible to picture these graphically without using a convenient *scale*. Thus, to picture 431,000 lb. we may let one division of the vertical axis represent 100,000 lb. Then 431,000 lb. is represented by 4.31, or approximately 4.3, of these divisions. Using this scale, the other numbers given above may be replaced by 4.7, 4.7, 4.1, 5.4, 6.3.

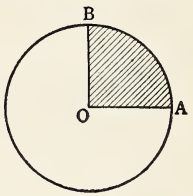
15. The value of the annual mineral production in Canada at ten-year intervals, expressed in millions of dollars, is as follows: 20, 60, 129, 210, 277. Represent these statistics by a graph.

16. The fur industry is of importance to Canada. Exports for certain years, in millions of dollars, are as follows:

1923	16.8	1927	18.9	1931	11.8
1925	15.4	1929	18.8	1933	10.4

Draw a graph representing these facts.

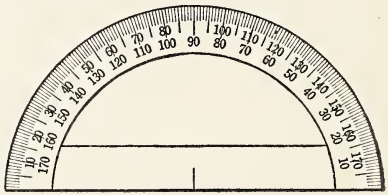
70. How to Make a Circle Graph. In making a circle graph, we must divide the circle into parts which represent the given items, by drawing lines from the centre of the circle. These parts are sometimes called *sectors* of the circle.



Thus, in the figure the radii OA and OB form a *sector* which represents one fourth of the circle.

A quick way of dividing a circle into the required parts is based on the use of a special instrument called the *protractor*. A form of the protractor is shown in the figure below.

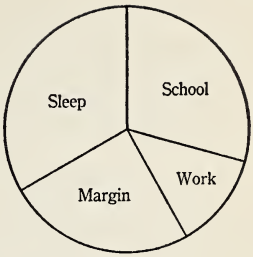
If two protractors of this kind are placed so as to form a circle, the circumference of this circle will show 360 equal divisions, or *degrees*. This is written 360°. By connecting each of these divisions with the centre of the circle, we obtain 360 small sectors. It is easy, therefore, to divide the circle into sectors that correspond to any desired fraction or decimal of 360°. By using 120° we obtain one third of the circle. Similarly, 90° is one fourth of the circle, 108° is 30 per cent of the circle, and so on.



Illustrative Example. The pupils of a health class were asked to prepare a "time budget" showing how they spent their time on a certain day. Each pupil was requested to find out how large was the "margin of time" left over when allowance was made for sleep, school hours, and other items.

William Smith reported that he spent his day as follows: sleep, 8 hours; school, 7 hours; outside work, 3 hours; margin, 6 hours.

In making a circle graph based on this time budget, he arranged his work as follows to obtain the sector necessary for each item:



	Number of Hours	Fractions of Day	Degrees in Required Sector
Sleep	8	$\frac{1}{3}$	120°
School	7	$\frac{7}{24}$	105°
Work	3	$\frac{1}{8}$	45°
Margin	6	$\frac{1}{4}$	90°
Total	24	$\frac{24}{24} = 1$	360°

Exercises. *Making Circle Graphs*

1. In a class of 36 pupils the following marks were given at the end of a month: A, 4 pupils; B, 9 pupils; C, 12 pupils; D, 8 pupils; and E, 3 pupils. Show by a circle graph how the marks of the class were distributed.

2. A family budget of $2400 was divided as follows: rent, $600; food, $700; clothing, $300; operating expense, $300; savings, $100; miscellaneous, $400. Show by a circle graph how the income of this family provided for the items listed above.

3. A study of the ages of pupils in a certain school showed that 70% were of normal age, while 20% were over-age pupils and 10% were under-age pupils. Draw a circle graph picturing this distribution.

4. The total expenditures of a large power company during a recent year were divided as follows: labor, 21%; fuel, 19%; material, 14%; taxes, 11%; depreciation, 9%; interest, 8%; dividends, 12%; surplus, 6%. Represent these items by a circle graph.

5. The "tax dollar" of a large city during a recent year was divided as follows: public instruction, 40¢; health, police, and fire departments, 20¢; public works, 17¢; debt service, 11¢; administration, 5¢; parks, 4¢; miscellaneous, 3¢. Show by a circle graph how the taxes of this city were used.

6. The cost of operating the average passenger automobile during a recent year was found by experts to be about 10 cents a mile. This cost was divided as follows: gas and oil, 18%; depreciation, 31%; maintenance, 12%; tires, 10%; licence, 6%; garage, 8%; interest and insurance, 15%. Draw a circle graph picturing these items.

7. Canada has been called "The Granary of the World." During a recent year the world exports of wheat were 584,000,000 bushels. These exports were divided as follows: Canada, 41%; Argentina, 22%; Australia, 21%; United States, 3%; other countries, 13%. Draw a circle graph picturing this distribution.

8. The total receipts from the Dominion income tax, as reported by the Department of National Revenue during a recent year, were derived as follows: individual taxpayers, 52%; manufacturers, 21%; transportation and public-utility companies, 6%; financial corporations, 8%; natural-resources corporations, 5%; other corporations, 8%. Draw a circle graph based on these figures.

9. Canada is the second largest gold producer in the world. In a recent year the world production amounted to $516,000,000, divided as follows: Africa, 50%; Canada, 12%; United States, 9%; rest of North and South America, 7%; Europe, 11%; Asia, 6%; Oceania, 5%. Draw a circle graph based on these figures.

10. In 1934 the estimated population of Canada was almost 11,000,000, divided approximately as follows: Prince Edward Island, 1%; Nova Scotia, 5%; New Brunswick, 4%; Quebec, 28%; Ontario, 32%; Manitoba, 7%; Saskatchewan, 9%; Alberta, 7%; British Columbia, 7%. Draw a circle graph showing these facts.

2. Graphs of Formulas

71. Picturing a Formula. It was shown in a preceding chapter that it is often possible to obtain a formula from a table. By reversing the process, it is possible to change a formula back into a table by substituting in the formula certain numerical values for the letters appearing in the formula. From such a table we may readily obtain a picture of the formula by using the graphic method.

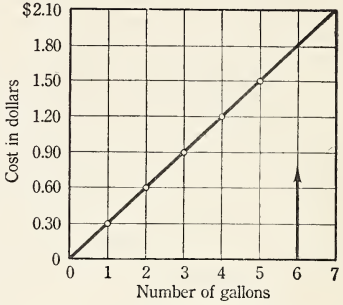
Illustrative Example 1. *The Cost Graph.* Suppose that gasoline costs 30¢ a gallon. Then the cost in dollars (c) of any number of gallons (g) is given by the formula

$$c = 0.30\, g.$$

By substituting in this formula successive values of g, we may make a table of corresponding values, as follows:

If $g =$	0	1	2	3	4	5	···
then $c =$	0	0.30	0.60	0.90	1.20	1.50	···

These corresponding sets of numbers can be pictured on squared paper, as here shown. On the horizontal line we record the number of gallons, and on the vertical line we indicate the cost. Thus, above 1 we locate a point opposite 0.30; above 2 we locate a point opposite 0.60; and so on. By connecting these points we obtain a straight-line graph.



Such a graph can be used as a "ready reckoner." That is, the cost of any number of gallons can be read directly from the graph if the graph is extended sufficiently far. Thus, to find the cost of 6 gallons, we locate the number 6 on the horizontal axis. The vertical line drawn from that point meets the graph at a point which is opposite $1.80. This is the required answer.

Similarly, if we wish to find how many gallons can be bought for a given sum of money, we locate first the given sum of money on the vertical axis. The horizontal line from that point meets the graph at a point which is directly above the desired number on the horizontal axis.

In other words, this cost graph does all the computing, or "reckoning," which would ordinarily have to be done by actual multiplication. Much time may therefore be saved by using the graph of a formula.

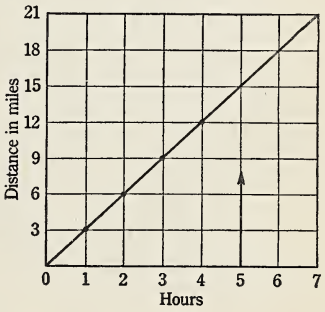
Illustrative Example 2. *The Uniform-Motion Graph.* Suppose that a man walks away from his home at the uniform

rate of 3 miles an hour. Then, after 2 hours of walking, he will be 6 miles from his home; after 3 hours, 9 miles; and so on. Evidently, the distance which he covers in any number of hours (h) may be obtained from the formula $d = 3\,h$.

The table below shows a number of corresponding values of h and d:

If $h =$	0	1	2	3	4	5	···
then $d =$	0	3	6	9	12	—	···

Using the plan explained above, we obtain the following *motion graph*. Observe that one space on the vertical axis represents 3 miles.



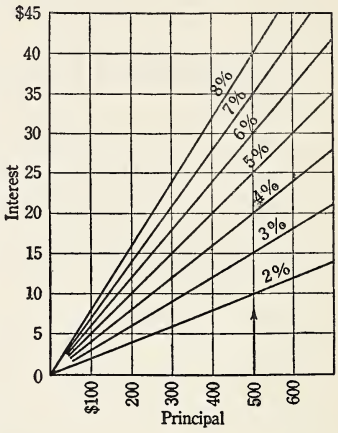
Hence, to find the distance covered in 5 hours, locate 5 on the horizontal axis. On the vertical line drawn from that point locate the point on the graph which will enable you to find the corresponding distance. It is situated opposite 15. Similarly, if the distance is known, we can find from the graph the corresponding number of hours.

Illustrative Example 3. *The Graph for Simple Interest.* In a previous chapter you studied the interest formula $i = prt$. If a sum of money is put at interest for *one* year, at the rate of 4 per cent, the formula becomes

$$i = .04\,p.$$

Thus, if $p = \$100$, $i = \$4$; if $p = \$500$, $i = \$20$; and so on.

In the same way we might make a formula for any other rate of interest. For 5 per cent we have $i = .05\,p$; for 7 per cent, $i = .07\,p$; and so on. By using the method explained above, we may then construct a set of graphs for any desired values of r, as suggested by the adjoining diagram.



If these graphs were extended sufficiently far, we could use them to find the interest on any convenient sum of money, at the rates represented in the diagram.

Thus, the interest on \$500 at 2 per cent is \$10; at 3 per cent, \$15; at 4 per cent, \$20; at 8 per cent, \$40.

Exercises. *Graphs of Formulas*

1. Using the method explained above, draw a graph of the formula $d = 10\,h$.

2. If the cost of a certain article is 8 cents a pound, the cost of any number of pounds is represented by the formula $c = 8\,n$. Draw a graph of this formula.

3. Draw a graph of the interest formula $i = .05\,p$, if p represents the principal and i the interest, the rate being 5 per cent.

Find from the graph (1) the value of i when $p = \$250$; (2) the value of p when $i = \$30$.

4. If the price of gasoline is 31 cents a gallon, show by a graph the cost of any number of gallons up to 10 gallons.

(1) From the graph determine the price of 6 gallons; of 9 gallons.

(2) How many gallons can be bought for \$2.17?

5. The formula for the perimeter of a square is $p = 4s$, where s represents the length of one side. Draw a graph of the formula.

From the graph find (1) the value of p when $s = 5\frac{1}{2}$; (2) the value of s when $p = 30$.

6. Draw a graph of the circumference formula $C = \pi d$.

SUGGESTION. Represent C on the vertical axis and d on the horizontal axis. Let $\pi = 3\frac{1}{7}$, and find the values of C for different values of d.

7. A man earns \$7.50 a day. Draw a graph showing his wages after 1 day, 2 days, 3 days, and so on.

8. A pupil's savings from week to week, in cents, were expressed by the formula $s = 40 + 6w$. Explain. Draw a graph of the formula. Determine by the graph how long it will take to save one dollar.

9. Construct a graph enabling you to read off the number of feet in a given number of yards. (Use the formula $f = 3y$.)

10. Make a graph by which you may find the number of centimetres in a given number of inches. (One inch = 2.5 centimetres, approximately.)

11. Show graphically the relation between pounds and kilograms. (One kilogram = 2.2 pounds, approximately.)

NOTE. Observe that when we draw the graph of a mathematical formula, we are really picturing in each case the relationship between changing quantities, which the formula expresses.

3. The Graphic Solution of Problems

72. Solving Problems Graphically. So far you have solved numerical problems either by ordinary arithmetic or by means of a table or by using the equation. There are problems, however, which can be solved more conveniently by using the *graphic* method. A few simple illustrations will show how graphs are used in solving problems and how this graphic method compares with the other methods.

Illustrative Example 1. Two neighbors, A and B, start on a trip at the same time and travel in the same direction. A travels at the rate of 20 miles an hour, while B's rate is 25 miles an hour. After how many hours will they be 15 miles apart?

Solving by Arithmetic. Since B covers 5 miles more than A each hour, it is evident that the two men will be 15 miles apart after 3 hours.

Solving by Table. By making a table showing the progress of A and B from hour to hour, we readily see that it will take 3 hours before the two men are 15 miles apart.

Hours	1	2	3	4	5	6
Distance of A . . .	20	40	60	80	100	—
Distance of B . . .	25	50	75	100	125	—

Solving by the Equation Method. Representing the required number of hours by h, we see that

$25\,h$ = the distance covered by B,

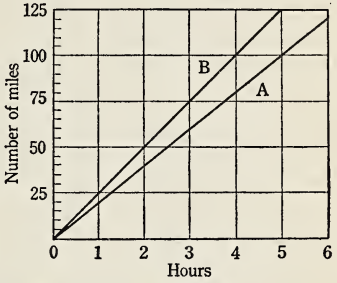
while $20\,h$ = the distance covered by A.

Hence we have $25\,h - 20\,h = 15$.

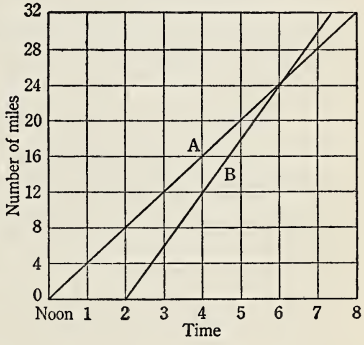
That is, $5\,h = 15$,

and $h = 3$.

Solving by the Graphic Method. The two straight lines in the diagram show the distances covered by A and B, respectively, from hour to hour. An inspection of the figure shows that after 3 hours the two men are 15 miles apart.



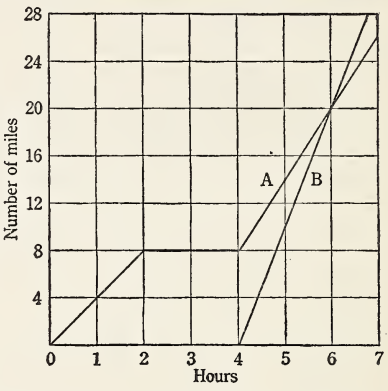
Observe that by either arithmetic or the equation method we obtain a definite answer to a *particular* question, while the table and the graph enable us to solve a *group of related problems.* The graphic method is more *general,* however, than the tabular method, since we can read at once from a graph the answers to many questions not suggested by the table. Thus, in the example above, how far apart will the men be after $4\frac{1}{2}$ hours?



Illustrative Example 2. A starts from his home at noon and walks at the rate of 4 miles an hour. B starts from the same place at 2 P.M., going in the same direction at the rate of 6 miles an hour. After how many hours will B overtake A, and how far from the starting point?

The diagram shows that B overtakes A in 4 hours, and 24 miles from the starting point.

Illustrative Example 3. A and B start at the same place, going in the same direction. A travels at the rate of 4 miles an hour, for 2 hours. Then he rests for 2 hours. He continues his trip at the rate of 6 miles an hour. B starts 4 hours after A and travels at the rate of 10 miles an hour. After how many hours will he overtake A, and how far from the starting point?



Can you interpret the graphs? What is the meaning of the horizontal portion of A's graph? What answers are suggested by the graphs?

Exercises. *Solving Problems by Graphs*

1. A grocer has two kinds of oranges which sell at 40 cents and 50 cents a dozen respectively. Make a cost graph for each kind, using the same axes for both graphs. From these graphs find out how many more oranges of the first kind can be bought for \$4 than of the second kind.

2. Henry saves 50 cents a week, while John saves 75 cents a week. Draw graphs representing the savings of each boy from week to week. From these graphs find out how soon John will have saved \$2 more than Henry.

3. Mary saved \$2 a week. Jane was given \$8 on her birthday and then she was able to save \$1 each week.

Draw graphs showing how much each girl had from week to week. How soon will the two girls have the same amount of money?

4. James saved 80 cents a week. His brother William started 4 weeks later, saving at the rate of $1.20 each week. Find out by means of graphs after how many weeks the savings of the two boys will be the same.

5. Mr. Smith bought a new automobile for $2400. If its value decreased at the rate of $400 a year, show by a graph the value of the automobile from year to year, for 5 years.

6. A storage tank contained 1000 gallons of gasoline. If 50 gallons are used each day, show by a graph the number of gallons remaining from day to day.

7. Two men, A and B, travel along the same road, in the same direction. A walks 4 miles an hour and B rides at the rate of 10 miles an hour. They start from the same place, but A starts 3 hours earlier than B. Show graphically how soon B will overtake A.

8. Two cities, A and B, are 350 miles apart. A train leaves A and travels toward B at the rate of 40 miles an hour. At the same time a train leaves B and travels toward A at the rate of 30 miles an hour. Determine graphically after how many hours the two trains pass each other.

SUGGESTION. On the vertical axis mark off a distance representing 350 miles. Call the extremities of this distance A and B. Then one train starts at A and the other at B.

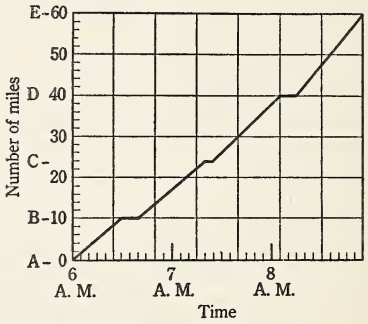
9. A and B start from the same place and travel in the same direction. A travels at the rate of 10 miles an hour. B starts 2 hours later. At what rate must B travel in order to overtake A in 4 hours?

10. Graphs may be used to picture a railroad time-table, as shown in the diagram below.

Study the graph and tell what the horizontal lines mean. If a second train left station A at 7 o'clock, its graph could be drawn by using the same axes. Similarly, if a train left station E and travelled toward A, its graph could be drawn by laying off the miles in the opposite direction.

Miles	Station	Time Schedule
0	A	Lv. 6 A.M.
10	B	Arr. 6.30 A.M.
		Lv. 6.40 A.M.
24	C	Arr. 7.20 A.M.
		Lv. 7.25 A.M.
40	D	Arr. 8.05 A.M.
		Lv. 8.15 A.M.
60	E	Arr. 8.55 A.M.

11. Secure a railroad time-table and show graphically the progress of all passenger trains in each direction between two stations, from 6 A.M. to 6 P.M.



12. Two men, A and B, start at the same time and travel along a certain road in the same direction toward the same town. A starts 24 miles away and walks at the rate of 4 miles an hour. B starts only 21 miles away and travels at the rate of 3 miles an hour. Show graphically the progress of the two men, and determine when one overtakes the other.

13. An algebra pupil found that it took him 15 minutes to study certain rules and 2 minutes to do one of the exercises following the rules. At that rate, show graphically

how much time he would require to complete the lesson for any number of exercises up to 15.

14. A printer finds that it costs him \$3 to set up the type for a school "program" and that he can then print the program at the rate of 100 copies for 50 cents. He decides to charge \$1 for a hundred copies. Draw a graph showing the actual cost of any number of copies *to the printer*. Also, using the same axes, draw a graph of the selling price, for any number of copies. How many hundred copies must be ordered in order that the printer may "break even"?

15. (*For recreation.*) Suppose that a ship leaves Montreal for Liverpool each day at noon; also, that a ship leaves Liverpool for Montreal each day at noon. If it takes exactly 7 days to cross the ocean in either direction, make a graphic chart showing the progress of these ships for a period of 2 weeks. In that case, how many ships coming from Liverpool will pass a particular ship which started from Montreal?

73. Summary. In this chapter you have learned

(1) How to make and interpret statistical graphs, such as bar graphs, broken-line graphs, curved-line graphs, and circle graphs.

(2) That graphs have many important uses, such as the following:

- **(*a*) They tell a statistical story at a glance.**
- **(*b*) They enable us to make comparisons.**
- **(*c*) They enable us to tell in advance the probable trend of events.**
- **(*d*) They often call attention to conditions that can be improved or corrected.**

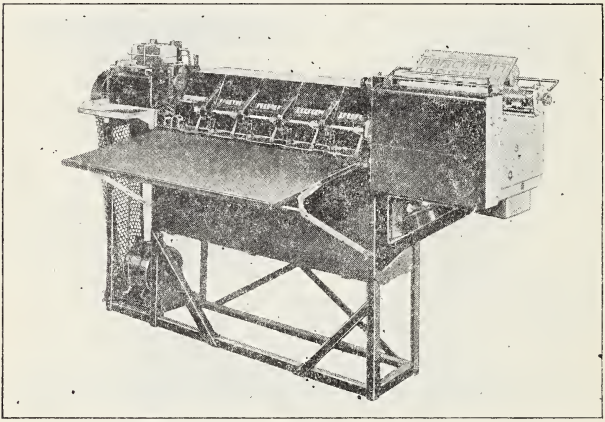
(3) How to make and use the graph of a formula.

(4) How to solve problems graphically.

Supplementary Honor Work

1. Make a collection of graphs found in newspapers and periodicals. Display these on a large sheet of cardboard and bring them to the classroom.

2. Make a graphic poster representing facts of interest to your school. You may use items relating to attendance, enrolment, games, school grades in different subjects, honor roll, bank deposits, health campaigns, and the like.



A modern tabulating machine which quickly sorts out thousands of statistical items with the aid of specially punched cards

3. Try to find out how a tabulating machine is used in sorting out quickly a large number of statistical items.

4. Read an account explaining how statistics are collected and recorded by the Dominion Bureau of Statistics.

5. Make a study of the scales which are used on graphic charts to picture very large numbers.

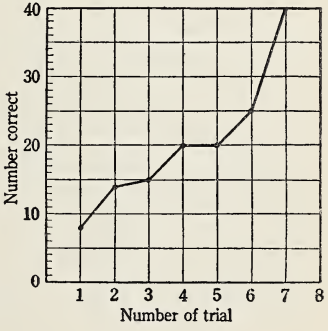
References

ELDERTON, W. P. and E. M. Primer of Statistics. A. and C. Black, London, 1910.

KING, W. I. The Elements of Statistical Method. The Macmillan Company, New York, 1915.

Test on Chapter IV

1. An algebra class was given a group of 40 practice exercises which were to be completed in 6 minutes. The accompanying graph shows a pupil's progress in 7 attempts to make a perfect score.



(1) What was his highest score?

(2) What was his lowest score?

(3) On which trial did he show no improvement?

(4) Which trial resulted in the greatest improvement of his score?

2. If butter costs 50 cents a pound, make a graph showing the cost of any number of pounds.

3. The normal weights of Canadian boys and girls at various ages are as follows:

Ages in years	9	10	11	12	13	14	15	16	17	18
Weight of boys (in pounds)	59	65	70	77	85	95	107	121	125	128
Weight of girls (in pounds)	57	62	69	78	89	98	106	112	113	114

Draw a graph showing the weights of *either* boys *or* girls.

4. Draw a graph of the formula $p = 4w + 5$, using in succession the following values of w: 0, 1, 2, 3, 4, 5, 6.

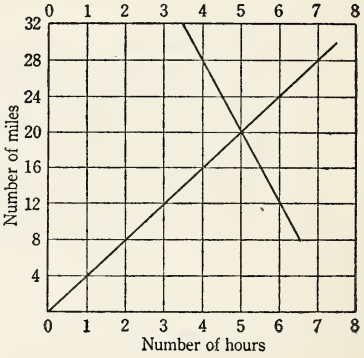
5. Represent graphically the related values given in the following table:

If r is . . .	0	1	3	4	7	9	10
then d is . . .	1	4	10	13	22	28	31

(1) From the graph find the value of d when $r = 5$.

(2) From the graph find the value of r when $d = 19$.

6. The accompanying graph tells the following story: Two boys start at different times from two places which are 32 miles apart, and travel toward each other. One boy walks at the rate of 4 miles an hour. The other boy starts $3\frac{1}{2}$ hours later, riding a bicycle at the rate of 8 miles an hour.



Study the graph and answer the following questions:

(1) How many hours after the second boy starts will the two boys meet?

(2) When they meet, how many miles will each boy have travelled?

CHAPTER V

SIGNED NUMBERS

1. Everyday Uses of Signed Numbers

74. Representing Numerical Changes. An algebra class took a test on equations. The same test was given again a week later, in order that the improvement of the class in solving equations might be measured. The records of six of the pupils are shown in the following table:

Pupil	First Score	Second Score	Comparison of Scores
Adams, John	95	97	+2
Brown, Mary	94	95	+1
Carroll, Henry	92	90	−2
Davis, Jane	90	87	−3
Edwards, Grace	86	89	+3
Farnum, William . . .	80	75	−5

You will observe that John Adams did *better* on the second test by 2 points. This is indicated by writing +2 in the third column of the table. Similarly, Mary Brown made a gain of 1 point. Hence her improvement is represented by +1. Henry Carroll, however, "went down" on the second test. He *lost* 2 points. This is indicated by writing −2 in the third column. A similar interpretation may be given for the remaining cases in the table. Try to explain why an improvement, or a gain, is represented by a *plus* sign, while a loss is indicated by a *minus* sign.

75. Signed Numbers. You will now be able to understand the following definitions.

Positive number. A number preceded by a plus sign is called a positive number.

Thus, $+4$, $+7$, and $+10$ are positive numbers.

Negative number. A number preceded by a minus sign is called a negative number.

Thus, -5, -4, and -7 are negative numbers.

Positive and negative numbers are often called *signed numbers.* Since they are used very commonly in algebra, they are also called *algebraic numbers.*

76. Measuring the "Ups and Downs" of Everyday Life. The world in which we live is a world of *changes.* Many values continually *fluctuate.* Like the waves of the sea they rise and fall. This is true, for example, of the hourly or daily temperatures in the course of a year. It is also true of wholesale and retail prices, of exports and imports, of business profits and losses, of census figures, crop statistics, taxes, stock-exchange quotations, attendance figures, and innumerable other items. In all these cases it is often found convenient to indicate the nature of the *fluctuation* or change by a corresponding signed number. Positive and negative numbers are used quite commonly for that purpose. The following exercises are intended to give you practice in recording changes by means of signed numbers.

Exercises. *Using Signed Numbers*

1. The table at the top of the next page shows the record of Richard and Howard in five tests. How does Richard's record compare with Howard's in each of these tests?

Test	I	II	III	IV	V
Howard	60	80	65	70	90
Richard	70	60	75	70	86
Comparison . . .	+10				

2. The number of baseball games won by each of six teams in two successive years is shown in the following table. What improvement did each team show?

Team	I	II	III	IV	V	VI
First year	40	30	20	15	10	14
Second year	42	26	25	14	10	17
Improvement	+2					

3. During a railroad strike the retail prices of certain commodities, in cents, changed as follows:

	Butter	Eggs	Oranges	Lettuce	Potatoes	Sirloin
Old price . .	65	60	75	10	100	40
New price . .	80	70	100	15	150	45
Change . . .	+15					

In each case indicate by a signed number the exact nature of the change in price.

4. The number of men employed by a big corporation during the first six months of two successive years was as follows:

Month . . .	January	February	March	April	May	June
First year . .	4320	5672	6423	6240	5420	4500
Second year .	4400	5700	6549	6032	5060	4027
Comparison. .	+80					

Show by signed numbers how the employment figures of the two years compare.

5. Study the following table, which is based on a report of the Toronto Stock Exchange:

Stocks	Last Quotation	Net Change
Bell Telephone	$148\frac{1}{2}$	$+2$
Brazilian Traction	$15\frac{1}{4}$	$+\frac{1}{8}$
British American Oil	26	$+\frac{1}{2}$
Canadian Pacific Railway	$13\frac{1}{2}$	$-\frac{1}{2}$
Consolidated Smelters	270	-3
Imperial Oil	$24\frac{1}{4}$	$+\frac{1}{4}$
International Nickel	$50\frac{1}{2}$	$+1$
United Gas	$12\frac{3}{8}$	$-\frac{1}{8}$
Winnipeg Electric Preferred	$14\frac{1}{4}$	$+\frac{7}{8}$

In the last column are found numbers preceded by the sign $-$ or $+$. The meaning is obvious. For example, in this case "$+2$" means that certain shares gained, or went up, \$2 per share, and "$-\frac{1}{2}$" means that certain other shares lost, or went down, $\$\frac{1}{2}$ per share since the publication of the previous report. How many of the stocks listed above show a gain?

77. Absolute Value. From the illustrations given above it may be seen that a signed number, such as $+4$, may be interpreted as meaning "a *gain* of 4," while -4 means "a *loss* of 4." In either case the number of units gained or lost is the same, namely 4.

This is indicated by saying that $+4$ and -4 have the same *absolute value*. The *absolute value of a signed number*, or its *numerical value*, is its value regardless of its sign. Observe that to each positive number there corresponds a negative number of the same absolute value.

The ordinary numbers of arithmetic are, of course, always regarded as positive. Thus, 6 has the same meaning as $+6$. Moreover, it is customary to read signed numbers

as they would be read in arithmetic. Thus, $+6$ is usually read "plus 6," and -6 is read "minus 6."

78. The Relation of Positive and Negative Numbers. If a man earns \$40 a week and then pays out the same sum to meet his expenses, it is clear that he has nothing left of his weekly earnings. Let us try to represent that fact by signed numbers.

Money which we earn is *added* to our resources, while money which we pay out is *subtracted* from our resources. Hence earnings are represented by *positive* numbers, while expenses are indicated by *negative* numbers. In the above illustration an income represented by $+$ \$40 is offset by expenses represented by $-$ \$40. The *combined* effect of having an income of \$40 and expenses of \$40 is zero. This may be written in brief form as follows:

$$(+40) + (-40) = 0.$$

It is easy to see that this relation applies to *any* positive number and its *corresponding* negative number. That is,

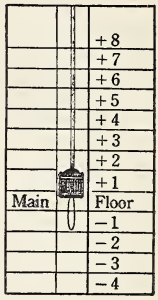
$$(+n) + (-n) = 0.$$

Hence we have the important fact that a *positive number plus the corresponding negative number equals zero.*

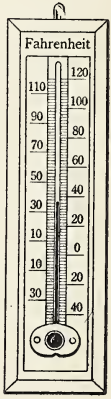
For this reason, positive and negative numbers having the same absolute value are said to be *opposites.* They destroy, or cancel, or neutralize each other. This is the reason why signed numbers may be used in many situations involving opposites.

79. Picturing Signed Numbers. The fact that signed numbers are used to measure "ups and downs" and to designate opposites also suggests a variety of simple methods of *picturing* signed numbers. Some of these will become clear from the following illustrations:

Illustrative Example 1. Let the figure suggest an elevator shaft. The horizontal partitions indicate the successive floors of the building. Let these floors above or below the main floor be numbered 1, 2, 3, and so on, as shown. An elevator starting from the main floor may make its trips in either of two directions. If it goes up to the first floor *above*, it has reached its goal by a "rise of 1"; if it goes down to the first floor *below*, it has reached its goal by a "drop of 1." Hence we may indicate the successive floors *above* the main floor by consecutive *positive* integers, and those *below* the main floor by consecutive *negative* integers. Thus, with reference to the main floor, the fifth floor up is numbered $+5$, and the second floor down is numbered -2.

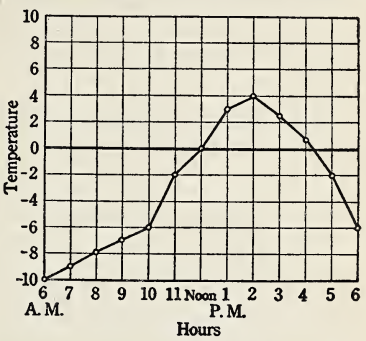


Illustrative Example 2. This is exactly the idea which underlies the number scale of a *thermometer*. The zero point on a thermometer evidently corresponds to the street level or the main floor mentioned above. Everybody knows the meaning of such expressions as "5 above zero" or "10 below zero."



Signed numbers are commonly used to designate temperatures above or below zero. Thus, $-10°$ means a temperature of 10 degrees below zero, while $+10°$ means 10 degrees above zero. In the case of temperatures above zero, the positive sign is usually omitted. With this agreement we can readily distinguish between $8°$ and $-8°$.

This method of designating temperatures is of importance in making and interpreting temperature graphs such as the one shown in the adjoining diagram.



What was the temperature at 6 A.M.?

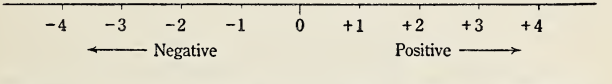
When was the temperature zero?

During which hours did the temperature rise?

During which hour did it begin to fall?

What is the highest temperature shown in the graph?

Illustrative Example 3. A number scale such as the one which is marked off on a thermometer may be in any desired position. In case of a horizontal line, it is customary to regard as positive the distances which extend to the *right* of the zero point, and as negative the distances which extend to the *left* of that point, as is shown in the following diagram. Such a diagram is called *a scale of signed numbers.*



Evidently it is constructed by marking off equal divisions, on a line of unlimited length, both to the right and to the left of some starting point, and numbering the divisions as shown. Fractions, either positive or negative, would be represented by points located between the integral division points.

A number scale may be used to explain not only the fundamental rules of signed numbers but also important applications based on these numbers. Thus, distances extending to the *east* or *north* are regarded as positive, while distances extending to the *west* or *south* are considered negative.

Since signed numbers are used to designate opposite *directions*, they are often called *directed numbers*.

Exercises. *Applications of Signed Numbers*

1. An elevator starting at the ground floor goes up to the tenth floor and then descends 6 floors. Where is it then? Draw a diagram showing its motions and express its final position in terms of signed numbers.

SUGGESTION. $(+10)+(-6)=+4$.

2. An elevator starts from the ground floor. It goes up 20 floors and then down 15 floors, then up 3 floors and then down 10 floors. Draw a diagram showing its motions and express its final position in terms of signed numbers.

3. A motor boat can make 25 miles an hour in still water. If it must make headway against a current that has a velocity of 10 miles an hour, how fast is the boat advancing? Explain.

4. An airplane, when flying at the rate of 100 miles an hour, encounters a head wind that has a velocity of 40 miles an hour. At what rate is the airplane advancing? Explain.

5. On a certain day Mr. Smith was notified by his bank that his chequing account showed deposits of \$240, but withdrawals of \$260. What was his "balance"?

6. What is meant by a balance of $-$ \$10?

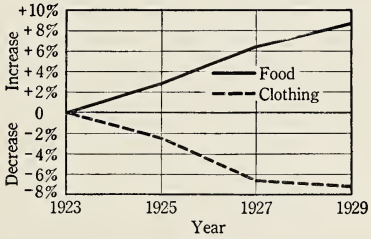
In each of the following exercises draw a graph showing the changes in temperature for the indicated period of time:

7. 7 A.M., $-6°$; 8 A.M., $-3°$; 9 A.M., $-2°$; 10 A.M., 0°; 11 A.M., 2°; noon, 7°; 1 P.M., 8°; 2 P.M., 8°; 3 P.M., 7°; 4 P.M., 5°; 5 P.M., 2°; 6 P.M., 1°.

8. 6 A.M., 8°; 8 A.M., 9°; 10 A.M., 12°; noon, 15°; 2 P.M., 16°; 4 P.M., 8°; 6 P.M., 0°; 8 P.M., $-4°$; 10 P.M., $-7°$; midnight, $-10°$.

9. Noon, 10°; 2 P.M., 9°; 4 P.M., 6°; 6 P.M., 5°; 8 P.M., 3°; 10 P.M., 2°; midnight, 0°; 2 A.M., $-2°$; 4 A.M., $-5°$; 6 A.M., $-10°$; 8 A.M., $-7°$; 10 A.M., 1°; noon, 4°.

10. Signed numbers are often used to show changes with reference to some value assumed as a standard of comparison. This is illustrated by the graph given below, which shows changes in the food and clothing prices from 1923 to 1929. The *zero line* represents the prices of 1923. Observe that six years later food prices had risen to a point



Changes in food and clothing prices

nearly 9% above the prices of 1923. Can you interpret the graph? What happened to clothing prices during the period considered?

11. The average attendance at a certain school during one year was 490. What was the actual attendance from day to day, during a certain week, if the fluctuation of the attendance, *with reference to that average*, was represented by these numbers: $+3, -4, +5, +7, -21$? Draw a graph showing these changes. Let the *zero line* represent the average attendance.

12. In a certain city the average tax rate per \$1000, during a period of 20 years, was found to be \$20.90. During the following five years the rate changed as follows: \$20.80, \$21.10, \$22.40, \$21.00, \$20.85. Express the fluctuation of the rate with reference to the average rate, and make a graphic picture of these changes.

2. Adding and Subtracting Signed Numbers

80. Addition. A few simple examples will serve to explain the addition of signed numbers.

1. If a man first gains \$5, and then also gains \$3, his total gain is \$8.

That is, $(+5) + (+3) = +8.$

2. If a man first loses \$5 and then loses \$3, his entire loss is \$8.

That is, $(-5) + (-3) = -8.$

3. If a man gains \$5 and then loses \$3, there is still a resulting gain of \$2.

That is, $(+5) + (-3) = +2.$

4. If a man first loses \$5 and then gains \$3, there is a resulting loss of \$2.

That is, $(-5) + (+3) = -2.$

Hence we see that in the addition of two signed numbers these four cases may arise:

(1) $(+5)+(+3)=+8.$

(2) $(-5)+(-3)=-8.$

(3) $(+5)+(-3)=+2.$

(4) $(-5)+(+3)=-2.$

81. Rules for Addition. From the four illustrations given above we infer the following rules:

1. *To add two signed numbers having like signs, find the sum of their absolute values and prefix to this sum their common sign.*
2. *To add two signed numbers having unlike signs, find the difference between their absolute values and prefix to it the sign of the number which has the greater absolute value.*

NOTE. The sum of two signed numbers is often called their *algebraic sum.*

Exercises. *Addition*

Add the numbers in each of the following cases:

1.	2.	3.	4.	5.	6.
$+10$	-20	-40	-100	$+7\frac{1}{2}$	-10.14
-5	$+15$	-50	$+80$	$-5\frac{1}{2}$	8.25

7. Complete the following statements:

(1) The sum of two *positive* numbers is always a _____ number.

(2) The sum of two *negative* numbers is always a _____ number.

(3) The sum of a positive and a negative number is *either* positive *or* negative, according as the _____ number or the _____ number has the greater absolute value.

8. Add $+5$ to each of the numbers in the following line:

$$+4, +6, +7, +10, +20, +47, +19.$$

To each of the numbers in the line above add -10.

9. Add -6 to each of the numbers in the following line:

$$-7, -2, -1, -12, -20, -41, -53.$$

To each of the numbers in the line above add $+12$.

10. A man deposits \$100 in a bank and later cheques out \$60 against his account. Express his balance as the sum of two signed numbers.

Solution. $(+100) + (-60) = +40$. Explain.

11. If a man has \$210 in the bank and through an oversight cheques out \$215, express his balance as the sum of signed numbers. What is meant by "overdrawing an account"?

12. The assets of a bank were \$20,000,000 during a certain year, while its liabilities were \$17,500,000. Express its net resources in terms of two signed numbers.

13. A firm failed in business. Its assets were \$100,000, but its liabilities were \$144,000. State its financial condition in terms of signed numbers.

Add the following:

14.	15.	16.	17.	18.	19.
$+2$	-6	-15	-20	$+16$	-1
$+7$	-3	$+10$	$+17$	$+7$	$+2$
-5	$+5$	-5	$+5$	-20	$+1$
-2	$+1$	$+6$	$+6$	-40	-2

EXPLANATION. To add *several* signed numbers, we usually find it convenient to add all the positive numbers, then to add all the negative numbers, and, finally, to find the sum of these two results.

Add in each of the following cases:

20. $+7, -4, +3, -2, +8, -10.$

21. $-40, 60, -80, 100, -200.$

22. Combine $(+10) + (-5) + (-4) + (-10) + (-9).$

23. Combine $(-7) + (-2\frac{1}{2}) + (+7\frac{1}{2}) + (-10).$

Find the missing number in each case:

24. $(+7) + (\quad) = -3.$ 27. $(\quad) + (-2) = +6.$

25. $(-9) + (\quad) = +10.$ 28. $(\quad) + (-5) = -11.$

26. $(+10) + (\quad) = 0.$ 29. $(\quad) + (-4) = 0.$

30. A man deposits \$300 in a bank and later cheques out \$250. He then deposits \$50 more and cheques out \$80. Express his balance as the sum of signed numbers.

31. A man has real estate worth \$10,000, and bank deposits amounting to \$9000. There is a mortgage of \$5000 on his property, and he has other debts amounting to \$4500. Express his net resources in terms of signed numbers.

Practice Exercises

The following table is to be used for practice in adding signed numbers, according to the directions given on the next page:

	I	II	III	IV	V	VI	VII
A	+8	−5	+10	−9	−20	+100	+7
B	+7	−4	−6	+7	+15	−1	−7
C	−4	+3	−2	−5	+19	+2	−1
D	−14	−15	+25	−19	−13	−5	+20
E	+1	−10	−3	+16	−4	+1	−10

DIRECTIONS. (1) *Add the numbers of row A to the corresponding numbers of row B.*

(2) *Add the corresponding numbers of rows B and C; of rows C and D; and so on.*

(3) *Add the corresponding numbers of rows A, B, and C; of rows B, C, and D; and so on.*

(4) *Add the corresponding numbers of column I and column II; of columns II and III; and so on.*

82. Subtraction. Let the points *A*, *B*, and *C*, in the diagram, represent three towns on a straight road. *B* and *C* are 5 miles and 7 miles, respectively, from *A*. In that case

the distance separating *B* and *C* can be found by *counting backward* from *C* to *B* or by *counting forward* from *B* to *C*. The result is the same in both cases, namely, 2 miles. That is, $7 - 5 = 2$.

Again, suppose that you buy 81¢ worth of groceries and give the grocer \$1. He then makes change by saying "81, 85, 90, \$1," while laying down 4¢, 5¢, and 10¢. Thus he finds the value of \$1 − 81¢ by counting *forward from the subtrahend* (81¢) *to the minuend* (\$1). In this business method of subtracting we may think of the *subtrahend* (81¢) as the *starting point*, while the *minuend* (\$1) represents the *goal*.

Illustrative Example 1. How does an elevator get from the fifth floor above the ground floor to the seventh floor above the ground floor? Evidently by going up two floors.

That is, $(+7) - (+5) = +2$.

Illustrative Example 2. To go from the seventh floor above to the second floor below, an elevator must go down nine floors.

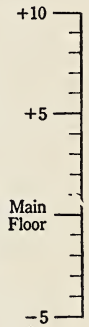
That is, $(-2) - (+7) = -9$.

Illustrative Example 3. If an elevator has to go from the second floor below to the seventh floor above, it must go up nine floors.

That is, $(+7) - (-2) = +9$.

Illustrative Example 4. To go from the second floor below to the fourth floor below, an elevator must go down two floors.

That is, $(-4) - (-2) = -2$.



That is, *in subtracting one signed number from another, we may find the difference by counting from the subtrahend to the minuend. If we count in the positive direction, the result is positive; but if we count in the negative direction, the result is negative.*

Oral Exercises. *Subtraction*

Perform the following subtractions by the "goal method" explained above; that is, use the subtrahend as the starting point and the minuend as the goal.

1.	3.	5.	7.	9.	11.
20	−15	−60	−41	−1	$+4\frac{1}{2}$
10	+10	−30	16	+1	$-2\frac{1}{2}$
2.	**4.**	**6.**	**8.**	**10.**	**12.**
+8	−40	20	17	−18	−19.13
−4	−50	100	−2	−13	+11.14

13. A man compares his bank balance at the end of each month with the corresponding balance of the preceding year and enters both in two parallel columns as shown in the table. In each case find the difference between the corresponding balances.

	This Year	Last Year	Difference
(1)	+ 100	+ 80	----
(2)	− 40	− 20	----
(3)	+ 50	− 10	----
(4)	− 2	+ 80	----

Subtract in each of the following cases:

14. $(+7)-(+5)$. 16. $(-2)-(+2)$. 18. $(-8)-(-2)$.

15. $(+5)-(-2)$. 17. $(+1)-(-1)$. 19. $(-20)-(+20)$.

20. Which number, when diminished by -7, equals $+6$?

21. The noon temperatures for one week were 17°, 12°, 18°, 4°, −2°, 0°, 5°. Find the difference between any two consecutive noon temperatures.

22. An airplane occupied in succession the following elevations: 1000 feet, 800 feet, 1200 feet, 1600 feet. Find the difference between any two consecutive elevations.

83. Relation between Addition and Subtraction. Compare the following sets of additions and subtractions:

(1) $(+7)+(-5)=+2$.	(1) $(+7)-(+5)=+2$.
(2) $(-7)+(-5)=-12$.	(2) $(-7)-(+5)=-12$.
(3) $(+7)+(+5)=+12$.	(3) $(+7)-(-5)=+12$.
(4) $(-7)+(+5)=-2$.	(4) $(-7)-(-5)=-2$.

Hence it is clear that subtracting $+5$ gives the same result as adding -5. Similarly, subtracting -5 gives the same result as adding $+5$. These relations will become clearer by considering that taking away an income of \$5 is the same as causing a loss of \$5. Also, if someone cancels a

debt of \$5 which you owe, he has really made you a present of \$5. In general,

$$-(+n) = +(-n),$$

and

$$-(-n) = +(+n).$$

That is, *subtracting a signed number is the same as adding the corresponding opposite number.*

84. Rule for Subtraction. You will now be able to understand the following rule:

To subtract one signed number from another, change the sign of the subtrahend and then proceed as in addition.

This method of subtracting is called the *reversal* method, since we change, or *reverse*, the sign of the subtrahend from $-$ to $+$, or from $+$ to $-$. In practice we may use either the goal method or the reversal method explained above. In the latter case, the change, or *reversal*, of sign is to be performed only *mentally*.

Thus, $(+8) - (-4)$ may be changed *mentally* into $(+8) + (+4)$.

Exercises. *Subtraction*

Perform the following subtractions by the reversal method:

1.	THINK:	4.	7.
$+8$	$+8$	-20	-12
$+4$	plus -4	$+10$	7
	$+4$		
2.	**THINK:**	**5.**	**8.**
-10	-10	-10	12
$+2$	plus -2	-10	-5
	-12		
3.	**THINK:**	**6.**	**9.**
-5	-5	-1	-1
-3	plus $+3$	$+7$	$+1$

10. A man had a balance of -100 dollars at the end of a certain month. A month later his balance was $+200$ dollars. How much better off was he then?

11. On a certain day the average temperature in Calgary was 12°. The following day it was $-2°$. What was the difference in temperature for these two days?

Perform the following subtractions:

12. $(+20) - (-10)$.
13. $(-40) - (+20)$.
14. $(-50) - (-40)$.
15. $(-1\frac{1}{2}) - (+3)$.
16. $(+7\frac{1}{2}) - (-5)$.
17. $(-100) - (+50)$.
18. $(-6.5) - (+3.4)$.
19. $(+17.14) - (-13.03)$.
20. Combine $(+10) + (+8) - (-2) + (-3) - (-4)$.
21. Combine $(-40) - (-20) + (+30) - (+30)$.

Find the missing numbers in each case:

22. $-7 - (\quad) = 2$.
23. $8 - (\quad) = -1$.
24. $-10 - (\quad) = 16$.
25. $-12 - (\quad) = 1$.
26. $40 - (\quad) = 20$.
27. $4 - (\quad) = 0$.
28. $6 - (\quad) = -10$.
29. $-8 - (\quad) = 12$.

30. Which number, when increased by -4, equals $+10$?

31. Which number, when diminished by $+5$, equals -7?

32. The sum of two numbers is -12. One of the numbers is 20. Find the other number.

33. The sum of two numbers is 40. One of the numbers is -100. Find the other number.

34. The sum of three signed numbers is -40. Two of the numbers are 4 and 25. Find the third number.

3. Multiplying and Dividing Signed Numbers

85. Multiplication. Consider the following preliminary explanation. If John is 12 years old now, 2 years ago his age was $12 - 2$, and 2 years from now it will be $12 + 2$. Hence it is customary to regard as *positive* a period of time coming *after* a certain fixed date, and as *negative* a period of time coming *before* that date.

Thus, dates in history are often expressed as coming after or before the birth of Christ. Instead of 500 A.D. and 300 B.C., one might use $+500$ and -300 respectively.

Illustrative Example 1. If a man's profit in business is \$4 a day, what will his profit be after 3 days? Evidently \$12.

That is, $(+3) \times (+4) = +12.$

Illustrative Example 2. If a man's expenses are \$4 a day, in 3 days his expenses will amount to \$12.

That is, $(+3) \times (-4) = -12.$

Illustrative Example 3. If a man's wages are \$4 a day, 3 days ago he had earned \$12 less than he has today.

That is, $(-3) \times (+4) = -12.$

Illustrative Example 4. If a man loses \$4 a day, 3 days ago he was \$12 better off than today.

That is, $(-3) \times (-4) = +12.$

86. Rule for Multiplication. From these illustrations we infer the following rule:

To multiply one signed number by another:

1. *Find the product of their absolute values.*
2. *Make the product positive if the two signed numbers have like signs; make the product negative if the two signed numbers have unlike signs.*

When *several* numbers are to be multiplied together, the multiplications may be performed in the order in which the numbers are written, or we may use a more convenient order.

Example 1. $(+5) \times (-4) \times (-3) = (-20) \times (-3) = +60.$

Example 2. $(-25) \times (-6) \times (-4) = (-25) \times (-4) \times (-6)$
$= (+100) \times (-6)$
$= -600.$

Observe that if the number of negative factors is *even*, the product is positive; if *odd*, it is negative.

Exercises. *Multiplication*

Find the product in each of the following cases:

1.	2.	3.	4.	5.	6.
-6	$+10$	-7	$+40$	$+2\frac{1}{2}$	-6.1
$+4$	-4	-5	-60	-4	-2.4

7. Multiply each of the numbers in the following line by $+3$:

$-6, \quad +7, \quad -1, \quad +10, \quad -12, \quad -20, \quad +5.$

8. Multiply each of the numbers in the preceding line by -2; by $+7$; by -1; by -10; by $+4$.

Perform the following multiplications:

9. $4 \times 15.$
10. $3 \times (-10).$
11. $(-5) \times 16.$
12. $(-10) \times 26.$
13. $(-19) \times (+20).$
14. $(-1.5) \times (+18).$
15. $17 \times (-12).$
16. $(+2.1) \times (-1.2).$
17. $(-6) \times (-20).$
18. $(-71) \times (-15).$
19. $(-1.2) \times (-10).$
20. $(-5.6) \times (-7.3).$

In the following exercises state the conditions of each problem by means of signed numbers:

21. A train is going east at the rate of 40 miles an hour. Where will it be 3 hours from now? Where was it 3 hours ago?

22. A train is going west at an average rate of 40 miles an hour. Where will it be 3 hours from now? Where was it 3 hours ago?

23. The temperature is rising 4° per hour. At that rate, what change in temperature will occur in 3 hours?

24. The temperature is falling 4° per hour. At that rate, what change in temperature will probably occur in 3 hours?

25. The shares of a certain stock suffered an average decline of 2 points a month during the last year. How does the present value of a share compare with its value 12 months ago?

26. The following table is based on reports of the Montreal Stock Exchange. The first column shows the number of shares sold on the day of the report. The last column gives the change in the value since the previous report. Compute the total gain or loss involved in the transfer of the shares listed below, no allowance being made for brokers' fees.

Sales	Name of Stocks	Net Change
190	Bell Telephone	$+\frac{3}{4}$
1820	Brazilian Traction	$+1\frac{1}{4}$
795	Canadian Pacific	$+1\frac{1}{2}$
100	Consolidated Smelters	-1
2080	International Nickel	$-\frac{1}{2}$
1870	Noranda	$+1\frac{1}{8}$
850	Steel of Canada	$+\frac{1}{4}$
90	Winnipeg Electric	$-\frac{1}{8}$

EXPLANATION. In writing the indicated product of two signed numbers, the multiplication sign may be omitted. Thus $(+4) \times (-5)$ may also be written $(+4)(-5)$.

Find the product in each case:

27. $(-25) \times (-20) \times 4 \times 5$.

28. $(-8) \times (-2) \times (-1) \times (+3)$.

29. $(-6)(+5)(-1)(-5)$.

30. $(+20)(-8)(+\frac{1}{2})(-5)$.

31. $(-2)(-2)(-2)(-2)$.

32. $(-1)(-1)(-1)(-1)$.

33. Name two numbers which might have -12 as their product.

34. If -2 is one factor of 4, what is the other factor?

35. If -1 is one factor of -3, what is the other factor?

36. If $+10$ is one factor of -20, what is the other factor?

87. Division. Division is the *inverse* of multiplication. For example, since $4 \times 3 = 12$, it is evident that $12 \div 4 = 3$. We may therefore derive the following statements from the corresponding cases in multiplication:

(1) $\frac{+12}{+4} = +3$.

(2) $\frac{-12}{-4} = +3$.

(3) $\frac{+12}{-4} = -3$.

(4) $\frac{-12}{+4} = -3$.

Check each of these results by multiplying the divisor by the quotient.

88. Rule for Division. From the above illustrations we infer the following rule of signs for division:

If two algebraic numbers have like signs, their quotient is positive; if they have unlike signs, their quotient is negative.

Exercises. *Division*

Numbers 1 to 12, oral

Perform the following divisions:

1. $(-100) \div (-4)$.
2. $(-40) \div (+10)$.
3. $(+35) \div (-7)$.
4. $(-2) \div (-1)$.
5. $(-4) \div (-4)$.
6. $(-9.1) \div (+1.3)$.
7. $18 \div (-2)$.
8. $49 \div (-7)$.
9. $(-60) \div 5$.
10. $(-100) \div 25$.
11. $(-15) \div (-5)$.
12. $(-2) \div (-2)$.
13. $(-50) \div 10$.
14. $10 \div (-50)$.
15. $40 \div (-8)$.
16. $8 \div (-40)$.
17. $12 \div (-4)$.
18. $(-4) \div 12$.
19. $(-12) \div 4$.
20. $(-12) \div (-4)$.
21. $(-60.8) \div 4$.
22. $(-18.4) \div (-.2)$.
23. $50 \div (-.02)$.
24. $(-5) \div (-5)$.

25. Which number, when divided by -7, equals 28?

26. Which number, when multiplied by -7, equals -28?

EXPLANATION. The *average* of a set of signed numbers is found exactly as in the case of the ordinary numbers of arithmetic.

Thus, the average of $+8$ and -4 is found by dividing the sum of these numbers by 2. That is, since $(+8) + (-4) = +4$, the average is $(+4) \div 2$, namely, $+2$.

27. Find the average of 10, -20, 15, -40, 100.

28. Find the average of -200, 460, -480, 1240.

29. During eight successive hours the temperature readings changed as follows from hour to hour at a certain place: 14°, 12°, 8°, 6°, 0°, $-2°$, $-4°$, $-10°$. What was the average hourly temperature during this period of time?

30. The noon temperatures, during a certain week, were 17°, 8°, 5°, 0°, − 2°, − 9°, − 2°. What was the average noon temperature?

31. Registration figures in five election districts showed the following changes when compared with the corresponding figures of the preceding year: − 42, + 12, − 32, + 57, − 40. What was the average gain or loss?

32. A number of business transactions resulted as follows: $1200 gain, $500 loss, $1000 gain, $200 gain, $800 loss. What was the average gain or loss?

EXPLANATION. When you find the average of a set of numbers, it is evident that some of the numbers must be larger than the average, while the remaining numbers must be smaller than the average. The amount by which any one of the numbers differs from the average is sometimes called its *deviation* from the average. Evidently, some of these deviations must be *positive*, while the others must be *negative*.

Thus, the average of 7, 12, 19, and 30 is 17. The deviations of these numbers from 17 are − 10, − 5, + 2, and + 13 respectively. Observe that the sum of these deviations is zero.

In general, it can easily be shown that *the sum of the deviations of a set of numbers from their average is zero.*

33. The following table shows the daily expenses of a pupil for one week, and the daily deviations from the average. Can you interpret each item?

DAY	CENTS SPENT	DEVIATIONS FROM AVERAGE	
1	13		− 31
2	27		− 17
3	45	+ 1	
4	34		− 10
5	81	+ 37	
6	17		− 27
7	91	+ 47	
Sum	308	+ 85	− 85
Average . .	$\frac{308}{7} = 44$	Check: (+ 85) + (− 85) = 0	

34. Find the average of 40, 39, 41, 38, 37, and 33. By following the method of Exercise 33, show that the sum of the deviations of these numbers from the average is zero.

35. Find the average of 50, 54, 45, 51, 45, 43, 61, and 67. Show that the sum of their deviations from the average is zero.

36. Find the average of 40, -24, 60, 100, -70, and -10. Show that the sum of their deviations from the average is zero.

Practice Exercises

Try to complete these exercises within 5 minutes.

Find the sum, the difference, and also the product in each of the following cases:

1. $+3$	6. $+7$	11. -4	16. $+2$	21. 0
$+5$	-4	0	0	-1
2. $+7$	7. $+9$	12. -6	17. -8	22. -5
$+5$	$+1$	-9	$+2$	-5
3. $+8$	8. -7	13. $+7$	18. -9	23. -1
-3	$+7$	$+9$	-3	$+9$
4. -4	9. -7	14. 0	19. $+1$	24. -7
$+7$	-3	-7	$+8$	0
5. -4	10. 0	15. $+6$	20. -9	25. $+5$
-1	-4	-6	0	-3

Find the following quotients:

26. $\frac{48}{8}$.	28. $\frac{-55}{-11}$.	30. $\frac{63}{-21}$.	32. $\frac{32}{4}$.	34. $\frac{-15}{3}$.
27. $\frac{24}{-6}$.	29. $\frac{-75}{15}$.	31. $\frac{56}{-7}$.	33. $\frac{-2}{+2}$.	35. $\frac{-1}{-1}$.

36. $\frac{-70}{-10}$. 39. $\frac{60}{-12}$. 42. $\frac{66}{-6}$. 45. $\frac{42}{-6}$. 48. $\frac{-9.6}{4}$.

37. $\frac{-18}{2}$. 40. $\frac{0}{17}$. 43. $\frac{-48}{-8}$. 46. $\frac{4.2}{-2}$. 49. $\frac{3}{-3}$.

38. $\frac{63}{7}$. 41. $\frac{-96}{4}$. 44. $\frac{72}{9}$. 47. $\frac{-8.1}{9}$. 50. $\frac{-36}{1.2}$.

89. **Summary.** In this chapter you have learned

(1) The meaning and important uses of positive and negative numbers.
(2) How to picture signed numbers.
(3) How to add signed numbers.
(4) How to subtract one signed number from another.
(5) How to multiply one signed number by another.
(6) How to divide one signed number by another.

Test on Chapter V

1. Find the sum, the difference, the product, and the quotient (upper number divided by lower number) in each of the following:

(1)	(2)	(3)	(4)
$+12$	-8	-24	12
-4	-16	$+6$	$+8$

2. Perform the indicated operations in each of the following:

(1) $(-15)+(+5)$.
(2) $(-12)+(-24)$.
(3) $(-10)-(+5)$.
(4) $(-25)-(-10)$.
(5) $(+8)\times(-4)$.
(6) $(-12)\times(-5)$.
(7) $(+300)\div(-20)$.
(8) $(+20)+(-8)+(+2)$.
(9) $(-5)+(-15)+(+8)+(-3)$.
(10) $12-7+(-3)+5-(-10)$.
(11) $(-3)\times(+5)\times(-2)$.
(12) $(-4)\times(-\frac{1}{2})\times(+5)\times(-6)$.
(13) $(-3.6)\div(-1.2)$.
(14) $(+8)\times(-5)\div(-4)$.

3. The table given below states the prices of some well-known types of automobiles during a certain year. The average price of these cars was \$867. By using signed numbers, indicate how the price of each type of car differed from this average.

(1) Coach \$822	(4) Coupé \$802
(2) Standard Sedan . . . \$911	(5) Sedan Delivery . . . \$878
(3) Coach with trunk . . \$845	(6) De Luxe Sedan \$944

4. The monthly gains or losses from a certain business, for six successive months, were as follows:

$+\$1250, -\$750, -\$350, +\$2450, -\$1000, +\$5000.$

(1) What was the total gain or loss for this period of time?

(2) What was the average gain or loss per month?

5. The following list suggests illustrations of signed numbers. Copy the numbers in the list and write after each a plus or a minus sign, according as the corresponding illustration refers to positive or negative numbers.

(1) Height above sea level.
(2) Deposits in a bank.
(3) A penalty in a football game.
(4) Motion upward.
(5) The forward motion of an airplane.
(6) North latitude.
(7) A drop in temperature.
(8) Income.
(9) Expenses.
(10) Dates B.C.
(11) A gain in a football game.
(12) The amount an airplane is retarded by wind.

CHAPTER VI

FUNDAMENTAL OPERATIONS

1. Addition and Subtraction

90. The Fundamental Operations. The methods of combining simple algebraic expressions by the four fundamental operations, namely, addition, subtraction, multiplication, and division, and the use of signed numbers in these operations, have been discussed in previous chapters. In this chapter the application of these principles to more complicated algebraic expressions will be explained.

91. Important Terms Defined. A knowledge of the following definitions will be necessary in connection with the work of this chapter.

Monomial. An algebraic expression whose parts are not separated by the signs + or − is called a *monomial*, or a *term*.

In the preceding pages you have become familiar with *simple* terms such as $3h$, $4n^2$, $5ab$, $\frac{h}{4}$. Expressions such as $3(a+b)$ and $\frac{b+c}{2}$ are sometimes called *compound* terms.

Monomials which have the same literal factors are called *similar terms*, or *like terms*.

Thus, $4ab$ and $-6ab$ are similar terms.

Any one of the factors of a term is called the *coefficient* of the product of the remaining factors.

Thus, in the term $4xy$, 4 is the coefficient of xy, x is the coefficient of $4y$, and so on.

In a term like $6\,a^2b$, the factor 6 is called the *numerical coefficient* of a^2b. Ordinarily the word "coefficient" is used in the sense of *numerical* coefficient. In any term whose numerical coefficient is not expressed, the coefficient 1 is understood.

Thus, ac, bcd, and x^2y are the same as $1\,ac$, $1\,bcd$, and $1\,x^2y$.

Binomial. An expression which consists of two terms is called a *binomial.*

Thus, $x + 2\,y$, $a^2 - b^2$, $3\,n + 6$, and $\frac{a}{b} + c$ are binomials.

Trinomial. An expression containing three terms is called a *trinomial.*

Thus, $2\,a + b - 5$, $x + \frac{y}{3} - z$, and $2(a + b) - 3\,c + d$ are trinomials.

Polynomial. Any expression consisting of two or more terms is called a *polynomial.*

Hence observe that the term "polynomial" really includes both the binomial and the trinomial.

92. Addition of Monomials. By making use of facts that have already been learned, the following additions can be easily understood:

$$\begin{array}{rrrr} +5\,d & -5\,d & +5\,d & -5\,d \\ \underline{+2\,d} & \underline{-2\,d} & \underline{-2\,d} & \underline{+2\,d} \\ +7\,d & -7\,d & +3\,d & -3\,d \end{array}$$

From these illustrative examples we may infer the following rule:

To add similar terms, find the algebraic sum of their coefficients and multiply it by the common literal factor.

NOTE. Observe that this statement expresses in more general form the Rule for Combining Similar Terms explained on pages 14 and 15.

Exercises. *Adding Similar Terms*

Numbers 1 to 12, oral

Add in each of the following cases:

1. $4n$, $14n$
2. $7w$, $-3w$
3. $18r$, $-12r$
4. $-15c$, $5c$
5. $12a$, $-8a$
6. $-32n$, $-17n$
7. $-27a^2$, $12a^2$
8. $-3bh$, $-13bh$
9. $-n^3$, $9n^3$
10. $10\frac{2}{3}ab$, $5\frac{1}{3}ab$
11. $-4\frac{3}{4}c$, $19\frac{1}{2}c$
12. $2.3b$, $-15.5b$
13. $-24s$, $-12s$, $6s$
14. $75d^2$, $-35d^2$, $15d^2$
15. $7ab$, $-4ab$, $-5ab$, ab
16. $-12e^3$, $8e^3$, $-4e^3$, $20e^3$
17. $19xy$, $17xy$, $-20xy$

Find the sum in each of the following cases:

18. $12n + (-4n)$.
19. $-15p + (+7p)$.
20. $-9s + (-3s)$.
21. $9n + (-3n) + (-2n)$.
22. $10c + (-12c) + (+c)$.
23. $-18s + (-2s) + (+5s)$.
24. $-11w + (+w) + (-7w)$.
25. $25w^2 + (-15w^2) + (+10w^2)$.
26. $12ab + (-3ab) + (+2ab)$.
27. $7n^3 + (-6n^3) + (-3n^3)$.

93. Subtraction of Monomials. As in the case of signed numbers, *a monomial may be subtracted from another monomial by changing the sign of the subtrahend and then proceeding as in addition.*

This is shown in the following illustrative examples:

Subtract:	(1) $+8\,d$	(2) $-8\,d$	(3) $-8\,d$	(4) $+8\,d$
	$\underline{+3\,d}$	$\underline{+3\,d}$	$\underline{-3\,d}$	$\underline{-3\,d}$
THINK:	$+8\,d$	$-8\,d$	$-8\,d$	$+8\,d$
	plus $-3\,d$	plus $-3\,d$	plus $+3\,d$	plus $+3\,d$
	$+5\,d$	$-11\,d$	$-5\,d$	$+11\,d$

Exercises. *Subtraction of Monomials*

Numbers 1 to 12, oral

Subtract in each of the following cases:

1. $5\,d$	5. $-10\,p$	9. $12\,n^2$	13. $-4\frac{3}{4}\,d$
$\underline{3\,d}$	$\underline{0}$	$\underline{-2\,n^2}$	$\underline{-2\frac{1}{4}\,d}$
2. $4\,w$	6. $5\,a$	10. $-9\,e^3$	14. $7\frac{2}{3}\,dw$
$\underline{7\,w}$	$\underline{-10\,a}$	$\underline{-e^3}$	$\underline{10\frac{1}{3}\,dw}$
3. $5\,n$	7. $-8\,b$	11. $-4\,bh$	15. $-20.3\,r^2$
$\underline{-2\,n}$	$\underline{b}$	$\underline{-14\,bh}$	$\underline{8.4\,r^2}$
4. $-6\,s$	8. c	12. 0	16. $4(d+b)$
$\underline{-4\,s}$	$\underline{-2\,c}$	$\underline{-3\,n}$	$\underline{-2(d+b)}$

Find the difference in each of the following cases:

17. $9\,a - (+5\,a)$.
18. $7\,c - (-3\,c)$.
19. $(-2\,b) - (+10\,b)$.
20. $(-12\,n) - (-2\,n)$.
21. $(-20\,s^2) - (-12\,s^2)$.
22. $14\,bh - (-7\,bh)$.
23. $dw - (-4\,dw)$.
24. $(-5\,e^3) - (+e^3)$.
25. Subtract $-24\,r^2$ from $-6\,r^2$.
26. How much larger is $25\,d$ than $-10\,d$?

94. Double Use of the + and − Signs. In the preceding pages the + and − signs have been used in two different senses. They either indicate opposite qualities such as

gain and loss, or they refer to the operations of addition and subtraction respectively. In any given case, it is usually not difficult to decide which interpretation is preferable. The following illustrative examples will show that either interpretation leads to the same result:

Illustrative Example 1. Find the sum of $+a$, $-b$, and $+c$.

The required sum might be written in the form

(1) $$(+a) + (-b) + (+c).$$

But we know that subtracting a signed number is the same as adding the corresponding opposite number. Hence the above expression might have been written

(2) $$(+a) - (+b) + (+c).$$

Since, however, a means $+a$, b means $+b$, and c means $+c$, the expression in step (2) may be written

(3) $$a - b + c.$$

Hence the signs which in the original example were *signs of opposition* have become *signs of operation.*

Illustrative Example 2. Combine $7n - 3n + 8n - 10n$.

By reversing the process explained above, we may obtain the result in either one of two ways. Thus, we may write

(1) $$(+7n) - (+3n) + (+8n) - (+10n),$$

in which case the original signs are regarded as *signs of operation.*

But the expression (1) may also be written in the form

(2) $$(+7n) + (-3n) + (+8n) + (-10n),$$

in which case the original signs appear as *signs of opposition.*

You will note that the same result is obtained in both cases, namely, $+2n$. It is usually better, however, to follow the *second* plan; that is, *to consider the given expression as the sum of a series of signed numbers.*

Exercises. *Double Use of + and − Signs*

Numbers 1 to 8, oral

In each of the following exercises you may obtain the result by considering the signs either as signs of operation or as signs of opposition. Combine the similar terms in each case:

1. $10 - 3 + 5$.
2. $5 + 2 - 9 + 12 - 7$.
3. $8 - 4 + 11 + 6 - 1$.
4. $2a + 9a - 3a$.
5. $5d - 4d + 6d - d$.
6. $7p + p - 3p - 4p$.
7. $12bh - 3bh - bh + 5bh$.
8. $10n^2 + 5 - 7n^2 + n^2 - 2$.
9. $8 - 10 + 3$.
10. $-10 + 4 - 2 - 7$.
11. $6 - 9 - 3 + 12$.
12. $-9 + 2 - 7 - 11 + 5$.
13. $3c - 5c + 8c - 12c$.
14. $b - 9b - 4b + 12b - 3b$.
15. $7n^2 - 8n^2 + 20n^2 - n^2 - 6n^2$.
16. $-5ab + 2ab - 11ab + ab + 4ab$.
17. $3x - 5 + 7x + 3 - 5x - 10$.
18. $10n - 3n + 8 + 7n - 16 + 2n$.
19. $-s^2 + 9s^2 - 15 - 5s^2 - 12s^2 + 9$.
20. $5a - 7a + 3b - 3a - b + 2a$.

95. Addition of Polynomials. Suppose that a man can travel m miles an hour when walking and d miles an hour when riding. One day he walks for 2 hours and rides for 3 hours, thus covering a distance of $(2m + 3d)$ miles. Another day he walks for 3 hours and rides for 5 hours, thus covering a distance of $(3m + 5d)$ miles. To find the total number of miles he travelled during these two days, we may arrange our work *horizontally*, as follows:

$$t = 2m + 3d + 3m + 5d = 5m + 8d.$$

Instead, we might have used a *vertical* arrangement, as follows:

$$\begin{array}{r} 2m + 3d \\ \underline{3m + 5d} \\ 5m + 8d \end{array}$$

Observe that we have thus added two *binomials*, which have been so arranged that *the similar terms are written under each other*. In like manner, we may add any polynomials.

96. Checking. In all mathematical work it is desirable to apply some check in order to make sure that the result obtained is correct. The method of checking which is found most useful in the case of the fundamental operations of algebra is that of substituting for each letter appearing in the given expressions some convenient numerical value. Thus, to check the addition of the three polynomials shown in the following illustrative example, we may let

$$a = 4, \quad b = 3, \quad \text{and} \quad c = 2.$$

Operation		*Check*
$5a + 4b + c$	$=$	$20 + 12 + 2 = 34$
$2a - 5b + 6c$	$=$	$8 - 15 + 12 = 5$
$\underline{-3a + 6b - 8c}$	$=$	$\underline{-12 + 18 - 16 = -10}$
$4a + 5b - c$	$=$	$16 + 15 - 2 = 29$

Observe that the sum of the numerical values of the three polynomials is 29, and that the numerical value of the final result is also 29. Hence we may infer that the work is correct.

Such a check cannot be accepted as final, however, for it may happen that, in substituting, two errors have been made such that one just offsets the other. Since this is quite unlikely to happen, the process explained above ordinarily affords sufficiently reliable evidence that the work is correct.

In checking expressions involving squares or cubes by this method, care should be exercised that the required operations are performed in their proper order (see pages 21 and 32).

Exercises. *Adding Polynomials*

Numbers 1 to 8, oral

Add in each of the following cases:

1. $3a + 10b$
   $5a + 6b$

2. $a + 4b$
   $3a + 7b$

3. $-x + 12$
   $4x - 5$

4. $7s^2 + 15t^2$
   $s^2 + 8t^2$

5. $3a + 2b - 5c$
   $a - 5b - 3c$

6. $x^2 + 6x + 9$
   $3x^2 + 2x + 1$

7. $4a^2 - 34ab + b^2$
   $2a^2 + 6ab - 3b^2$

8. $n^2 - 3n + 5$
   $5n^2 - 2n - 7$
   $-3n^2 + n + 2$

Add in each of the following cases, and check as explained on page 162:

9. $2a + 5b$
   $3a - 2b$

10. $7a + 15$
    $-4a + 5$

11. $-8n + 3a$
    $4n - a$

12. $9x - 12y$
    $-4x - 8y$

13. $-5x^2 + 11$
    $x^2 - 8$

14. $5a + 7b - 3c$
    $a - 2b - 5c$

15. $x^2 - 4x + 10$
    $-2x^2 - 6x + 2$

16. $3x^2 - x + 5$
    $-x^2 + 7$

17. $2a^2 - 3ab + 4b^2$
    $6a^2 - 2ab - 4b^2$

18. $r^2 + 3rs - 5s^2$
    $2r^2 + 7s^2$
    $-5r^2 - rs - 5s^2$

NOTE. Observe the arrangement of the terms in the preceding exercises. A similar plan should be used in the work in Exercises 19–24.

Find the sum of the polynomials given in each of the following cases, and check each result:

19. $4n + 3$, $n - 17$, $-2n + 8$.

20. $w^2 + 4w + 4$, $2w^2 - 3w + 5$, $-7w^2 + w - 12$.

21. $6h^2 - 3h + 15$, $3h^2 - 9$, $h + 3$.

22. $2x^2 - 5x + 7$, $x - 3x^2 - 8$, $9 + x^2 - 2x$.

23. $3a + 2b - 9c$, $c - 5b + a$, $7b - 6a + 2c$.

24. $10n^2 - 3n - 8$, $n - 4n^2$, $6 + n^2 - 5n$.

25. The dimensions of a rectangle are $2s + 1$ and $5s - 3$. A second rectangle has the dimensions $3s - 4$ and $10s - 25$.

(1) Find the perimeter of each rectangle.

(2) Find the sum of these perimeters.

97. Subtraction of Polynomials. In the subtraction of polynomials the work may be arranged as in addition.

For example, subtract $7a^2 - 5a - 3$ from $5a^2 + 7 - 3a$, and check the result.

Operation		*Check.* Let $a = 3$.
$5a^2 - 3a + 7$	=	$45 - 9 + 7 = +43$
$7a^2 - 5a - 3$	=	$63 - 15 - 3 = +45$
$-2a^2 + 2a + 10$	=	$-18 + 6 + 10 = -2$

Thus the problem becomes one of subtracting similar terms which have been written under each other.

Checking. The check might have been performed as in arithmetic by adding the resulting difference to the subtrahend and thus obtaining a sum equal to the minuend. The substitution check is used, however, since it gives a uniform method of checking for all operations.

NOTE. If the given polynomials contain squares or cubes, the number 1 should not be used in checking, since it will not detect a mistake in an exponent. For instance, if by mistake $-2a^4$ had been written

for $-2a^2$ in the above result, either term would have the value -2 when we let $a = 1$. Hence the result will appear to check in either case. If, however, another number is substituted for a, the result will not check when the mistake $-2a^4$ has been made. Thus, when $a = 3$, we have $-2a^4 = -162$, while $-2a^2 = -18$, and in this case the result will check only for the value of the correct term.

Exercises. *Subtraction of Polynomials*

Numbers 1 to 10, oral

Subtract in each of the following cases:

1. $5a + 9b$ / $7a + 3b$
2. $3a - 5$ / $-7a + 4$
3. $-2b - 12c$ / $3b - 7c$
4. $6x + y$ / $2x + 3y$
5. $-4x^2 + 14$ / $x^2 - 7$
6. $8e^3 - 27$ / $-4e^3$
7. $6bh$ / $-2bh + 4$
8. $7d - 5w + 3h$ / $2d - 5w - 2h$
9. $a^2 - 4a + 4$ / $-2a^2 - 8a + 14$
10. 0 / $3n^2 - 6n + 9$
11. $3dw + 2wh - 4$ / $-4wh + 8$
12. $5\frac{1}{2}a^2 - 7ab - 3\frac{1}{3}b^2$ / $2\frac{1}{2}a^2 + 3\frac{2}{3}ab - 1\frac{2}{3}b^2$
13. $5.7x^2 - 3x + 1.2$ / $x^2 + .8$
14. $8e^3 + e - 5$ / $-4e^2 + 6e - 5$

Perform the indicated subtractions and check each result:

15. Subtract $12a - 3b + c$ from $-8a + 3b - 2c$.
16. Subtract $-4p - 3r + 5t$ from $-4p + 6r - t$.
17. From $2r^2 + 7rs - s^2$ subtract $r^2 - 3rs + 2s^2$.

18. Subtract $7\,bh + 2\,b - h$ from $bh - 2\,b + 5\,h$.

19. Subtract $-3\,a - 11\,b + 7$ from 0.

20. Subtract $5\,x^2 - 2\,x$ from $2\,x^2 - 9\,x - 1$.

21. From $-15\,w^2 - 8\,wd + d^2$ subtract $5\,w^2 - 10\,d^2$.

22. How much larger is $c^2 - 12\,c + 20$ than $-5\,c^2 + c - 8$?

23. By how much does $e^2 - 15\,e + 2$ exceed $e^2 + 5\,e - 2$?

24. How much smaller is $-4\,A + 2\,B - 14\,C$ than $A - 6\,B - 7\,C$?

25. Subtract $3\,x^2 - 5\,x + 2$ from $4\,x^3 - 7\,x^2 + 6\,x - 3$.

FIG. 1

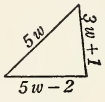


FIG. 2

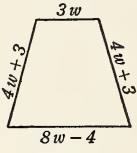


FIG. 3

26. How much larger is the perimeter of Fig. 1 than that of Fig. 2?

27. Subtract the perimeter of Fig. 2 from that of Fig. 3.

28. Subtract the perimeter of Fig. 3 from that of Fig. 1.

29. Subtract the perimeter of Fig. 2 from the sum of the perimeters of Figs. 1 and 3.

2. MULTIPLICATION

98. Rule of Signs for Multiplication. You have learned that in multiplication the product of two numbers having like signs is *positive*, whereas the product of two numbers having unlike signs is *negative*. This rule may be applied to literal as well as to numerical expressions.

Thus, $(+a) \times (+b) = +ab,$

$(-a) \times (-b) = +ab,$

$(+a) \times (-b) = -ab,$

and $(-a) \times (+b) = -ab.$

NOTE. The *rules* of algebra are often called its *laws*. Thus the *Rule of Signs* may also be referred to as the *Law of Signs*.

99. Rule of Order for Multiplication. The volume of a box whose dimensions are 2 feet, 3 feet, and 5 feet may evidently be found in several ways. Thus,

$V = 2 \times 3 \times 5 = 2 \times 5 \times 3 = 3 \times 2 \times 5$, and so on.

Hence we may infer the following Rule of Order:

In multiplication, the factors may be arranged in any order.

That is, in general form, $a \times b = b \times a$.

100. Rule of Grouping for Multiplication. To obtain the result from any one of the products indicated above, such as $2 \times 5 \times 3$, we may first multiply 2×5 and then multiply the product by 3, or we may multiply 2 by the product of 5 and 3. For example,

$$2 \times 5 \times 3 = (2 \times 5) \times 3 = 2 \times (5 \times 3).$$

Hence we may infer the following Rule of Grouping:

In multiplication, the factors may be grouped in any convenient manner.

That is, in general form,

$$a \times b \times c = (a \times b) \times c = a \times (b \times c).$$

Exercises. ***Applying the Rules of Order and Grouping***

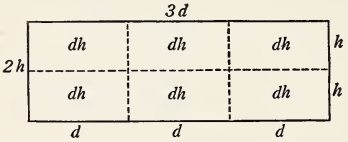
Find rapidly the following products:

1. $2 \times 14 \times 5$.
2. $13 \times 10 \times 3$.
3. $5 \times 7 \times 4 \times 5$.
4. $8 \times 5 \times 125$.
5. $16 \times 7 \times \frac{1}{2}$.
6. $15 \times 14 \times \frac{1}{3} \times 4$.
7. $12.5 \times 7 \times 8 \times 3$.
8. $1\frac{1}{4} \times 12 \times \frac{2}{3} \times 16$.

9. In finding the product of 7, 5, 8, and 2, there are 24 ways of arranging the factors. Write as many of these arrangements as you can find.

10. Find the area of a rectangle whose dimensions are $3\,d$ and $2\,h$.

Solution. From the diagram below it will be seen that the area is $6\,dh$.



The algebraic solution involves both the Law of Order and the Law of Grouping, as the following development will show:

$$A = 3\,d \times 2\,h.$$

This may be written $A = 3 \cdot d \cdot 2 \cdot h.$

Applying the Rule of Order, we have

$$A = 3 \cdot 2 \cdot d \cdot h.$$

Applying the Rule of Grouping, we have

$$A = (3 \cdot 2) \cdot (d \cdot h).$$

Hence $A = 6\,dh.$

11. Find the area of a rectangle whose dimensions are (1) $5\,l$ and $3\,w$; (2) $9\,b$ and $4\,a$; (3) $20\,d$ and $\frac{1}{2}\,h$.

Find the following indicated products:

12. $3 \times 4\,h.$

13. $5\,d \times 2.$

14. $8\,a \times 4.$

15. $2\,a^2 \cdot 6.$

16. $5 \cdot 4\,b \cdot 2.$

17. $2\,a \cdot 4\,b.$

18. $3\,a \cdot 5\,a.$

19. $6\,x \cdot 5 \cdot 2\,y.$

20. $2\,a \cdot 3\,b \cdot 4\,c.$

21. $5\,n \cdot 4\,n \cdot 2\,n.$

22. $2\,x \cdot (-5\,y).$

23. $(-3\,a) \cdot (-2\,b).$

24. $(-12\,h) \cdot \frac{1}{2}\,d.$

25. $8\,b \cdot (-2\,b).$

26. $3\,w \cdot (-7\,w) \cdot 4.$

101. Rule of Exponents for Multiplication. You have learned that

$$a \times a = a^2,$$

and that

$$a \times a \times a = a^3.$$

In like manner, $a \times a \times a \times a \times a = a^5$, which is read "$a$ to the fifth power." In the term a^5 the small number 5 is called an *exponent.* It indicates how many times a has been used as a factor. The factor a which is repeated is called the *base.* The result a^5 is called the *power.* Similarly, in the expression $2^4 = 16$, 2 is the *base,* 4 is the *exponent,* and 2^4, or 16, is the *power.*

When no exponent is written with a number or letter, the exponent 1 is understood. Thus, a means a^1, and 3 dw^2 means $3^1 \cdot d^1 \cdot w^2$.

Accordingly, to find the product of $a^3 \times a^2$, we have

$$a^3 \times a^2 = (a \cdot a \cdot a) \times (a \cdot a) = a \cdot a \cdot a \cdot a \cdot a = a^5.$$

In brief form, $a^3 \times a^2 = a^{3+2} = a^5$.

Hence *the exponent of the product of powers having like bases is the sum of the original exponents.*

That is, in general form, $a^m \times a^n = a^{m+n}$.

Likewise, $(a^2)^3 = a^2 \cdot a^2 \cdot a^2 = a^{2+2+2} = a^6$,

or, more briefly, $(a^2)^3 = a^{2\times 3} = a^6$.

In general, $(a^m)^n = a^{m\times n}$.

In like manner,

$$(ab)^2 = ab \times ab = a \cdot a \cdot b \cdot b = a^2b^2,$$

or, more briefly, $(ab)^2 = a^2b^2$.

In general, $(ab)^n = a^nb^n$.

NOTE. Distinguish carefully between a coefficient and an exponent. For instance, in 4 h, 4 is a coefficient; in h^4, 4 is an exponent. The expression 4 h means $h + h + h + h$, while h^4 means $h \cdot h \cdot h \cdot h$.

Exercises. *Exponents*

1. What is the meaning of x^4? b^3? a^7? a^5?

2. Using exponents, write each of the following expressions in brief form:

(1) $n \cdot n \cdot n \cdot n$.
(2) $s \cdot s \cdot s \cdot s \cdot s \cdot s$.
(3) $a \cdot a \cdot a \cdot a \cdot a$.
(4) $x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$.
(5) $2 \times 2 \times 2 \times 2$.
(6) $10 \times 10 \times 10$.

3. Distinguish between $5n$ and n^5.

4. Write, in brief form, $b + b + b$; also $b \cdot b \cdot b$.

5. Using coefficients and exponents, write in briefest form each of the following expressions:

(1) $x + x + x + x$.
(2) $yy + yy + yy$.
(3) $3\,aaa - 2\,aaa$.
(4) $2\,bb + 5\,bb - 3\,bb$.
(5) $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$.
(6) $5 \cdot 5 \cdot 5 \cdot 5$.
(7) $5\,rrr + 3\,rr$.
(8) $5\,rrr \times 3\,rr$.

6. What is another way of writing $(ab)^2$? $(mn)^3$? $(xy)^5$?

7. Express as a power of 2 each of the following numbers:

4, 8, 16, 32, 64, 256.

8. Express as a power of 5 each of the following numbers:

25, 125, 625.

9. Write one million as a power of 10.

Find the following indicated products:

10. $a^4 \cdot a^3$.
11. $n^2 \cdot n^7$.
12. $p^5 \cdot p^3$.
13. $x^8 \cdot x^4$.
14. $s^5 \cdot s^6$.
15. $n^3 \cdot n$.
16. $x^2 \cdot x^4$.
17. $a \cdot a^5$.
18. $b^4 \cdot b^2 \cdot b^3$.
19. $x^3 \cdot x \cdot x^2$.
20. $R^2 \cdot R^3$.
21. $(-d) \cdot d^4$.
22. $p^{10} \cdot (-p^5)$.
23. $(-n^9) \cdot (-n^{11})$.
24. $r^6 \cdot (-r^3) \cdot r^9$.

102. Multiplication of Monomials. The multiplication of monomials involves the rules of signs, order, grouping, and exponents, which have already been considered.

Thus, $(+3\,a^2b) \times (-2\,a^3b^2c) = (+3) \cdot (a^2b) \times (-2) \cdot (a^3b^2c)$
$= (+3) \cdot (-2) \cdot a^2 \cdot a^3 \cdot b \cdot b^2 \cdot c$
$= -6\,a^5b^3c.$

Hence we may use the following method of finding the product of two or more monomials:

1. *Find the product of the coefficients, applying the Rule of Signs for Multiplication.*
2. *Multiply this result by the product of the literal factors, giving to each letter in the product an exponent equal to the sum of its exponents in the separate factors.*

Exercises. *Multiplication of Monomials*

REMEMBER THE KEY WORDS: signs, order, grouping, exponents, and their abbreviations, S., O., G., E.

Numbers 1 to 10, oral

Perform the indicated operations in each of the following cases:

1. $2\,w^3 \cdot 3\,w^5$.
2. $3\,d^2 \cdot d^4$.
3. $n^4 \cdot (-n^3)$.
4. $(-8\,r^2)(-3\,r^5)$.
5. $(-5\,d)^3$.
6. $(-2\,a^2b^3)(6\,a^2b)$.
7. $(5\,D)(-3\,D^3)(-2\,D^2)$.
8. $(-b)(-10\,b^2)$.
9. $5\,a^7 \cdot (-10\,a^2)$.
10. $-(2\,p)^3$.
11. $(-10\,R^2)(-10\,R^2)$.
12. $5\,y(-3\,x^2y)$.
13. $(-3\,s^3t)^2$.
14. $-(3\,s^3t)^2$.
15. $c(8\,c^3)(-\frac{3}{4}\,c^5)$.
16. $(-2\,nd)(-3\,n)(-4\,d^2)$.

17. Find the square of each of the following:

$5, -4, n^4, 3a^3, -7d, -6b^5, a^2b, 5x^3y^2.$

18. Find the cube of each of the following:

$3, -2, c^2, 4n^3, -3s, +5d^4, -x^2y^6, 2cd^5.$

19. Show that the following statements are true:

(1) $2^4 = 4^2$. (3) $2^3 \cdot 2^4 = 2^7$. (5) $10^3 \cdot 10 = 10^4$.
(2) $8^2 = 4^3$. (4) $3^2 \cdot 3^3 = 3^5$. (6) $(2^3)^2 = 2^6$.

EXPLANATION. *Order of Operations.* The directions given previously concerning the evaluation of algebraic expressions may now be extended to include any indicated power. The operations should be performed in the following order:

1. *Find the numerical value of any indicated power.*
2. *Perform the indicated multiplications or divisions in the order in which they occur.*
3. *Perform the indicated additions or subtractions.*

When a parenthesis occurs in the original expression, the operations indicated within the parenthesis should be performed first.

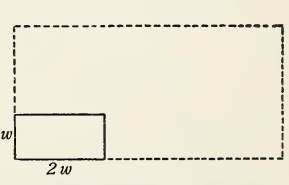
20. Find the numerical value of the following if $w = 2$ and $d = 3$:

(1) $5w^2 - 6d$. (4) $(2w)^3 - 2d^2$. (7) $(-3w^3)^2 + 2wd^2$.
(2) $10wd^2$. (5) $w - (-2d)^2$. (8) $5w^2d - (-3d)^3$.
(3) $4w^3d^2$. (6) $5w^3d$. (9) $6(w^3d)^2 + 5(-w)^5$.

21. The length of a rectangle is twice its width w.

(1) What is its perimeter? its area?

(2) If each side is made three times as long as in the original rectangle, what is the resulting perimeter? the resulting area?



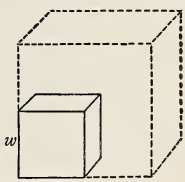
(3) Does the area become three times as large when each side is made three times as long?

22. A box in the form of a cube has an edge whose length is w.

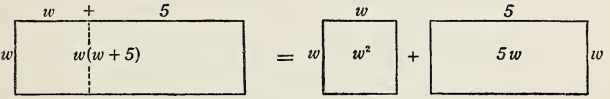
(1) What is the total length of its edges? the total surface? the volume?

(2) If each edge is doubled, what is the total surface of the resulting solid? the volume?

(3) Does doubling the edge double the surface? the volume?



103. Multiplication of a Polynomial by a Monomial. Suppose that the width of a rectangle is w, while the length is $w + 5$. To find the area of this rectangle, it becomes necessary to multiply w by $w + 5$. The diagram shows that the product is $w^2 + 5\,w$.



That is, $$w(w + 5) = w^2 + 5\,w.$$

Similarly, $2\,a(a - 2\,b + 3\,c) = 2\,a^2 - 4\,ab + 6\,ac.$

From these illustrations we infer the following rule:

To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial, uniting the resulting terms by their proper signs.

That is, $n(a + b - c) = na + nb - nc.$

This rule is frequently called the *Rule of Distribution for Multiplication.*

In applying this rule, the work may be arranged either *horizontally* or *vertically,* as shown on the next page.

Horizontal Arrangement

$$-3a(a^2-2a+1) = -3a^3+6a^2-3a.$$

Check. Let $a = 2$.

$$-6(4-4+1) = -24+24-6.$$
$$-6(1) = -6.$$
$$-6 = -6.$$

Vertical Arrangement		*Check.* Let $a = 2$.
a^2-2a+1	=	$4-4+1=+1$
$\underline{\quad -3a \quad}$	=	$\underline{\quad -6 \quad}$
$-3a^3+6a^2-3a$	=	$-24+24-6=-6$

Exercises. *Multiplying Polynomials by Monomials*

Numbers 1 to 10, oral

Find the indicated products:

1. $3(a+b)$.
2. $5(d-2w)$.
3. $2n(3x-y)$.
4. $-4r(r^2+2r-3)$.
5. $6(a-3b+5c)$.
6. $b(3r-2b)$.
7. $-2(-x+3y-6)$.
8. $5a(2b-6c)$.
9. $-3p(p^2-6p)$.
10. $n^2(3n^3+2n-1)$.

Find the indicated products and illustrate each by a drawing:

11. $d(d+3)$. 12. $h(2h+d+4)$. 13. $2h(h+2w+3d)$.

Find the product in each of the following cases, and check each result:

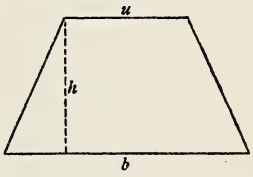
14. $y^2(2y-3)$.
15. $2n^2(n^2+3n-1)$.
16. $-a^3(a^2-3a+5)$.
17. $-h(1-12h+32h^2)$.
18. $5a-3b+7$
$\underline{\qquad\qquad 3a}$
19. n^2+5n-8
$\underline{\qquad\qquad -2n^2}$
20. $8y^3+4y^2-y+1$
$\underline{\qquad\qquad\qquad 5y^2}$
21. $7x^2-4xy+y^2$
$\underline{\qquad\qquad -3x^2y}$

22. Find the area of a rectangle whose length is $4d - 5$ and whose width is $3d$.

23. The altitude of a rectangle is $5h$. What is its area if its base is $10h + 3$? if its base is $12h - 7$?

24. What is the volume of a rectangular solid whose height is $3b - 2$ and the area of whose base is $12b^2$?

25. The adjoining figure represents a *trapezium*. The two parallel sides are called its *bases*. If the altitude of a trapezium is denoted by h, the lower base by b, and the upper base by u, its area is found by the formula $A = \frac{h}{2}(b + u)$. Rewrite the formula, performing the indicated multiplication.



26. If the bases of a trapezium are b and c respectively and its altitude is 4, what is its area?

27. Find the area of a trapezium having an altitude of 6 and having as its bases w and 10 respectively.

28. An airplane travelled $(50d + 12)$ miles an hour for $3h$ hours. How far did it fly?

Find rapidly the following products:

29. 1001×17.

SUGGESTION. $1001 \times 17 = (1000 + 1)17 = 17{,}000 + 17 = 17{,}017$.

30. 105×16.

31. 112×6.

32. 204×8.

33. 1006×14.

34. 2003×21.

35. 99×13.

36. 999×16.

37. 197×3.

38. 99×101.

104. Removing Parentheses. In multiplying a polynomial by a monomial, we had occasion to remove the parentheses enclosing the given polynomial, by applying the Rule of Distribution. Since this process of removing parentheses is used very commonly in algebra, additional practice in dealing with parentheses is desirable. The following explanatory statements should be kept in mind:

1. *The removal of parentheses requires the careful application of the Rule of Signs for Multiplication.*

Thus, $3(a+b-c) = 3a+3b-3c.$

Similarly, $-2a(a^2-3a+5) = -2a^3+6a^2-10a.$

2. *Any parenthesis without an indicated coefficient may be considered as having a coefficient of 1.*

Thus, $(a+b) = 1(a+b) = a+b.$

Also, $-(x-2y) = -1(x-2y) = -x+2y.$

3. *The accepted order of operations requires an indicated multiplication to be performed before an addition or a subtraction.*

Thus, $5-3(a+2b) = 5-3a-6b.$

Hence $5-3(a+2b)$ *does not equal* $2(a+2b).$

4. *If one set of parentheses occurs within another, it is usually advisable to remove them one set at a time, starting with the innermost.*

Thus,

$$-3c[a+2b(b-c)] = -3c[a+2b^2-2bc] = -3ac-6b^2c+6bc^2.$$

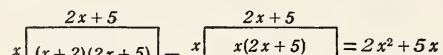
Exercises. *Removing Parentheses*

Remove the parentheses and combine similar terms:

1. $5(2a+3)-7.$
2. $2(x-3)+5.$
3. $5n-4(n+7).$
4. $-3(5d-4)+12.$

5. $4\,a(2\,a+b-c)+7\,ac$.
6. $3+5(2\,x-9)$.
7. $3\,b-2\,b(b-1)$.
8. $11\,x(x-4)+7\,x$.
9. $-7\,a(-3-2\,a)+4\,a$.
10. $16\,y^2+4\,y(3\,y-5)$.
11. $5(3\,n-4)+6\,n$.
12. $-(2\,x-y)-5\,y$.
13. $5\,b(-2\,m+b)+bm$.
14. $11\,a+2(3\,b-c-a)$.
15. $8\,w(w^2-5\,w-6)+25\,w^2$.
16. $8-4(-2\,x+3)$.
17. $6-(s-5)$.
18. $10\,w+w(6-4\,w)$.
19. $9\,n^2-6\,n+(1-4\,n+4\,n^2)$.
20. $mx+4(mx-3)+11$.
21. $7(5\,x-3)-6(3\,x+4)$.
22. $3(5\,a+1)+2(3\,a-5)$.
23. $5(x-y)-(2\,x+y)$.
24. $6\,s-3(r+s+t)+(2\,r-s+3\,t)$.
25. $5\,x-7\,y+(2\,x-y)$.
26. $-(8\,d-3)+(-3\,d+4)$.
27. $6\,n^2-(3\,n+2)+2\,n(n-5)$.
28. $5[2\,n-6(n+2)]-3\,n+8$.
29. $3\,a[6+2(a-3)]+10\,a^2$.
30. $s\{s-2(3-5\,s)+7\}-2\,s^2$.
31. $2\,x-3[2\,x-3(x-5)]$.
32. $4\,y+2\,y[3\,y-(5-y)+6]$.
33. $8\,r-[7-2(3\,r-5)]$.
34. $9\,x^2+3\,x+2[2\,x-5\,x(x-1)-10]-(2\,x^2+7\,x)$.
35. $(3\,x-5)-\{6\,x+4\,x(2\,x-7)+3\,x\}+12$.

105. Multiplication of a Polynomial by a Polynomial. If the dimensions of a rectangle are $2x + 5$ and $x + 2$, its area is found by multiplying $2x + 5$ by $x + 2$. The diagram shows how this product may be obtained.



That is, $(x + 2) \times (2x + 5) = x(2x + 5) + 2(2x + 5)$
$= 2x^2 + 5x + 4x + 10 = 2x^2 + 9x + 10.$

Thus we see that the product is found by a repeated application of the Rule of Distribution. Observe that each term of the first binomial was multiplied by each term of the second binomial.

The work may also be arranged vertically, as follows:

Operation		*Check.* Let $x = 2$.
$2x + 5$	=	$4 + 5 = 9$
$x + 2$	=	$2 + 2 = 4$
$2x^2 + 5x$		36
$+ 4x + 10$		
$2x^2 + 9x + 10$	=	$8 + 18 + 10 = 36$

NOTE. A similar process is used in the arithmetical multiplication of numbers having two or more places, except that in algebra the work is carried out from left to right. For example, multiply 43 by 26 in the ordinary way, and then multiply $(40 + 3)$ by $(20 + 6)$ by the method explained above, comparing the corresponding steps.

106. Arrangement of Terms. In $4x^3 + 6x^2 + 5x + 7$, the terms are said to be arranged in *descending order*, since the exponents of the letter x descend, from left to right, in the order of magnitude. Similarly, the terms of $7 + 5x + 6x^2 + 4x^3$ are said to be arranged in *ascending order*. In the multiplication of polynomials, it is usually

advisable to arrange the terms in either descending or ascending order, as shown in the following illustrative example:

Descending Order		Ascending Order
$-x^2+x+5$	or	$5+x-x^2$
$-7x+3$		$3-7x$

If more than one letter occurs in the given expressions, the terms may be arranged according to the powers of one particular letter. For example, $5x^3-xy^2+2y^3-3x^2y$, when arranged according to the descending powers of x, would read $5x^3-3x^2y-xy^2+2y^3$.

From the preceding explanations and illustrative examples we infer the following rule for multiplying a polynomial by a polynomial:

1. *Arrange the terms according to the descending or ascending powers of the same letter.*
2. *Multiply each term of the multiplicand by each term of the multiplier, and add the partial products.*

Exercises. ***Multiplying a Polynomial by a Polynomial***

Multiply:

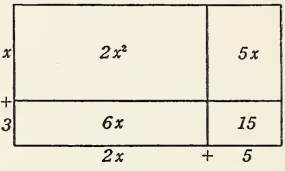
1. $x+3$ by $x+5$.
2. $x-12$ by $x+7$.
3. $x-8$ by $x-7$.
4. $n+15$ by $n-4$.
5. $2n+3$ by $3n+10$.
6. $12r+3s$ by $r-4s$.
7. $7x^2+2x$ by $x-3$.
8. $3c^2+2d$ by $3c^2+2d$.
9. $2r^2-9s^2$ by $2r^2+9s^2$.
10. $3w^2-6w$ by $2w+1$.

Multiply in each case and check the result:

11. $(x^2-3x+2)(x+3)$.
12. $(n-5+2n^2)(n-2)$.
13. $(5y^2+3-y)(3y-5)$.
14. $(3x^2-5x-2)(2x+7)$.

15. $(7 + 2n^2 - 3n)(5n - 9)$.
16. $(a + 5)(5a^2 - 8 + 3a)$.
17. $(3 - 10h)(6h - 4h^2 - 10)$.
18. $(a^2 + 2ab + b^2)(a + b)$.
19. $(c^2 + cd + d^2)(c - d)$.
20. $(r^2 - 2rs + 4s^2)(3r - s)$.
21. $(t^2 - 3s^2 + 4st)(s - 2t)$.
22. $(3x - 5 + x^2)(3x^2 + 7 - 2x)$.
23. $(2a^2 + 3 - 6a)(5a - a^2 - 4)$.
24. $(4n + 3n^2 - 2)(n^2 - 6 - 2n)$.
25. $(3d - 1)(2d + 2)(d - 1)$.
26. $(x^3 - 2x^2 + 8 + 4x)(x - 2)$.

27. The adjoining figure shows a rectangle whose dimensions are $x + 3$ and $2x + 5$.



(1) Find the area by actual multiplication.

(2) Check your result by means of the diagram. The sum of the areas of the four small rectangles equals the total area of the large rectangle.

(3) What is the area if $x = 2$? if $x = 5$? if $x = 10$?

28. The dimensions of a closed rectangular box are represented by $w + 1$, $2w + 7$, and $3w - 2$.

(1) Find the expression which represents its total surface.

(2) Find its volume.

29. The width of a rectangular coal bin is 1 foot less than its height, and its length is 3 feet less than twice its height.

(1) If h represents its height, what is its volume?

(2) If the bin is 9 feet long, what are the other dimensions? what is its volume?

30. If a train travels at the rate of $(2r + 5)$ miles per hour for $(n - 3)$ hours, what distance will it cover?

31. If a grocer sells $(6p - 1)$ pounds of coffee at $(c - 2)$ cents per pound, and $(5p + 3)$ pounds of tea at $(2c - 3)$ cents per pound, what is the total amount received?

3. Division

107. Rule of Signs for Division. You have learned that in division the quotient of two numbers having like signs is positive, while the quotient of two numbers having unlike signs is negative. This rule may be applied to literal as well as numerical expressions. Thus,

$$\frac{+20d}{+5} = +4d. \qquad \frac{-20d}{+5} = -4d.$$

$$\frac{-20d}{-5} = +4d. \qquad \frac{+20d}{-5} = -4d.$$

108. Rule of Exponents for Division. Suppose that a^5 is to be divided by a^3. It is evident that the quotient must be an expression which, when multiplied by a^3, produces a^5. Hence, since $a^3 \times a^2 = a^5$, we see that $\frac{a^5}{a^3} = a^2$. This result might have been obtained as follows:

$$\frac{a^5}{a^3} = \frac{\cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdot a \cdot a}{\cancel{a} \cdot \cancel{a} \cdot \cancel{a}} = a^2.$$

Since three of the factors of a^5 are removed when we divide by a^3, we may write, more briefly,

$$\frac{a^5}{a^3} = a^{5-3} = a^2.$$

Hence, in the case of powers having like bases, *the exponent of the quotient may be found by subtracting the exponent of the divisor from that of the dividend.*

That is, in general form, $\frac{a^m}{a^n} = a^{m-n}$.

109. Division of Monomials. The division of one monomial by another involves the above rules of signs and exponents.

Thus, $\frac{-18\,b^3h^4}{+6\,b^2h} = -3\,bh^3$, since $(+6\,b^2h)\cdot(-3\,bh^3) = -18\,b^3h^4$.

Also, $\frac{-24\,r^2s^5t}{-3\,r^2s^2} = +8\,s^3t$, since $(-3\,r^2s^2)\cdot(+8\,s^3t) = -24\,r^2s^5t$.

Hence we may use the following method of dividing a monomial by a monomial:

1. *Find the quotient of the coefficients, applying the law of signs for division; then multiply this result by the quotient of the literal factors.*
2. *Find the quotient of the literal factors by subtracting the exponent of each letter in the divisor from that of the corresponding letter in the dividend, thus obtaining the exponent to be used with that letter in the quotient. Factors which have like bases and exponents cancel each other and give 1 as a factor of the quotient.*

Exercises. *Division of Monomials*

Numbers 1 to 10, oral

Divide:

1. $12\,d$ by (-2).
2. $-18\,a$ by (-6).
3. $-24\,n$ by 4.
4. $-28\,t^2$ by 4.
5. $40\,lwh$ by (-5).
6. $-42\,nr$ by (-6).
7. $-32\,b^3$ by 8.
8. n^6 by n^2.
9. d^8 by d^5.
10. nd^3 by $(-d^2)$.
11. $-w^5$ by w.
12. $-15\,dw$ by $(-3\,w)$.
13. m^3n^2 by $(-n^2)$.
14. $-8\,a^3b^4$ by $4\,ab^3$.
15. $24\,s^4$ by $(-6\,s^3)$.
16. $-60\,a^3b^2$ by $5\,a^2b^2$.

Perform the indicated division in each case:

17. $\frac{-36\,r^2s^5}{-12\,r^2s}$.

18. $\frac{54\,b^2h}{-6\,bh}$.

19. $\frac{-75\,p^6n^3}{15\,p^2n}$.

20. $\frac{-35\,b^2xy^3}{-7\,bxy^2}$.

21. $\frac{40\,r^5s^3t^2}{-5\,r^2t^2}$.

22. $\frac{48\,x^3y^2z}{-3\,x^2y^2}$.

23. Divide each of the following by $3\,a^2$:

$6\,a^3$, $12\,a^5$, $15\,a^2$, $-21\,a^4b$, $18\,a^2b^3$, $-33\,a^3b^2c$.

24. Divide each of the following by $-4\,rs^3$:

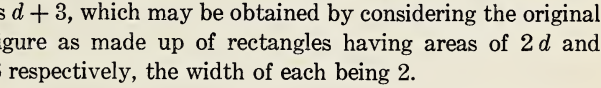
$8\,r^2s^3$, $-12\,r^5s^5$, $20\,rs^4$, $-36\,r^3s^3t$, $28\,rs^3$, $-4\,r^2s^6$.

25. Show that the following statements are true:

(1) $2^8 \div 2^3 = 2^5$. (3) $10^6 \div 10^4 = 10^2$.

(2) $3^5 \div 3^2 = 3^3$. (4) $5^3 \div 5^2 = 5$.

110. Division of a Polynomial by a Monomial. Suppose that the area of a rectangle is $2\,d + 6$, while its width is 2. To find the length of this rectangle it becomes necessary to divide $2\,d + 6$ by 2. The diagram shows that the result is $d + 3$, which may be obtained by considering the original figure as made up of rectangles having areas of $2\,d$ and 6 respectively, the width of each being 2.



That is, $$\frac{2\,d+6}{2} = \frac{2\,d}{2} + \frac{6}{2} = d + 3.$$

Similarly, $$\frac{2\,d^3 - 4\,d^2 + 8\,d}{2\,d} = d^2 - 2\,d + 4.$$

Such a procedure in division corresponds to the Rule of Distribution for Multiplication, and hence we may state the following rule:

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial, uniting the resulting terms by their proper signs.

That is, $\frac{na + nb - nc}{n} = \frac{na}{n} + \frac{nb}{n} - \frac{nc}{n} = a + b - c.$

Checking. Such examples may be checked either (1) by showing that *divisor* $\times$ *quotient* = *dividend*, or (2) by numerical substitution. The latter method is illustrated below.

Operation

$$\frac{-4n^3 - 12n^2 + 8n}{-4n} = n^2 + 3n - 2.$$

Check. Let $n = 2$.

$$\frac{-32 - 48 + 16}{-8} = 4 + 6 - 2.$$

$$\frac{-64}{-8} = 10 - 2.$$

$$8 = 8.$$

Exercises. *Dividing a Polynomial by a Monomial*

Numbers 1 to 7, oral

Divide:

1. $9n + 6$ by 3.
2. $4b + 12$ by 4.
3. $6n - 3$ by 3.
4. $n^2 - 2n$ by n.
5. $6n - 18n^2$ by $6n$.
6. $10a^2 - 25a$ by $(-5a)$.
7. $21x - 7$ by 7.
8. $27a^2 - 18a - 9$ by (-9).
9. $45x + 60x^2 - 20x^3$ by $5x$.
10. $12b^2 - 21bh + 18bh^2$ by $(-3b)$.

11. $3\,a^3 - 6\,a^2 + 18\,a$ by $3\,a$.

12. $9\,d^3 - 6\,d^2 + 3\,d$ by $3\,d$.

13. $x^4 - 3\,x^3 + x^2$ by x^2.

14. $4\,a^2 - 8\,ab + 4\,ab^2$ by $4\,a$.

Perform the indicated division in each case:

15. $\dfrac{6\,bh - 18\,b^2h - 12\,bh^2}{-\,6\,bh}$.

16. $\dfrac{-\,10\,x^3y + 6\,x^2y^2 - 2\,xy^3}{-\,2\,xy}$.

17. $\dfrac{3\,r^3s^3 - 5\,r^2s^2 + rs}{rs}$.

18. $\dfrac{6\,d^4b^3 - 12\,d^3b^2 - 9\,d^2b^3}{-\,3\,d^2b^2}$.

19. $\dfrac{16\,a^4b^2 - 24\,ab^3}{8\,ab^2}$.

20. $\dfrac{12\,m^5n^3 - 24\,m^4n^4 + 18\,m^2n^5}{6\,m^2n^3}$.

21. If a man's income for 12 days is $(36\,d - 24)$ dollars, how much does he earn per day?

22. An auto travelled $(150\,dh - 35\,h)$ miles in $5\,h$ hours. What was its hourly rate?

23. What is the length of a rectangle whose area is $4\,n^2 + 12\,n$ and whose width is $4\,n$?

24. The volume of a rectangular solid is $3\,h^3 - 6\,h^2 - 45\,h$. If its height is $3\,h$, what is the area of its base?

25. A rectangular solid has a base whose area is $5\,h^2$. If the volume is $5\,h^3 - 5\,h^2d + 15\,h^2$, find its altitude.

26. What is the altitude of a cylinder whose volume is $27\,\pi h + 45\,\pi - 9\,\pi d$ and the area of whose base is $9\,\pi$?

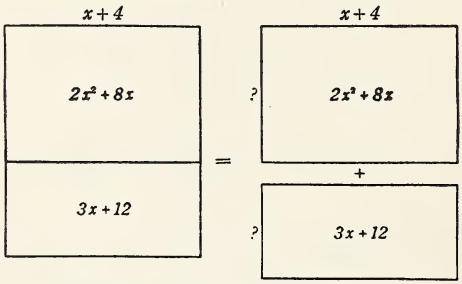
111. Division of a Polynomial by a Polynomial. It was shown previously that when $x + 4$ is multiplied by $2x + 3$, the process may be indicated as follows:

$$\begin{aligned}(2x + 3)(x + 4) &= 2x(x + 4) + 3(x + 4)\\ &= 2x^2 + 8x + 3x + 12\\ &= 2x^2 + 11x + 12.\end{aligned}$$

Hence it follows, from the principles of division, that

$$\frac{2x^2 + 11x + 12}{x + 4} = 2x + 3.$$

The actual process of division can readily be explained by means of a diagram such as the following:



Can you interpret the diagram? What is the width of the upper rectangle? How is it obtained? What is the width of the lower rectangle? Why?

The algebraic solution may be arranged in compact form as follows:

	Operation		*Check*
	$2x + 3$	(Quotient)	$x + 4$
(Divisor)	$x + 4 \overline{)\,2x^2 + 11x + 12}$	(Dividend)	$2x + 3$
	$2x^2 + 8x$	$[2x(x + 4)]$	$2x^2 + 8x$
	$3x + 12$		$+ 3x + 12$
	$3x + 12$	$[3(x + 4)]$	$2x^2 + 11x + 12$

EXPLANATION. The first term of the dividend is the product of the first term of the divisor and the first term of the quotient. Hence the first term of the quotient, in the example above, is found by dividing $2x^2$ by x. The result is $2x$. This term represents the width of the upper rectangle in the diagram. The product of $2x$ and $x+4$ represents the first partial product, $2x^2+8x$. Subtracting this from the dividend, we obtain the remainder, $3x+12$, which represents the area of the lower rectangle in the diagram. Dividing $3x$ by x, we obtain 3, the second term of the quotient, which represents the width of the lower rectangle. Multiplying 3 by $x+4$, we obtain the second partial product, $3x+12$, or the area of the lower rectangle. By subtracting this from $3x+12$, we find that there is no remainder. In other words, the area of the large rectangle has been shown to be the product of $x+4$ and $2x+3$. The work may then be checked by actual multiplication.

112. Arrangement of Terms. As in the case of multiplication, the terms of both polynomials should be arranged according to the *descending* or the *ascending* powers of some common letter, before the division is begun. This is shown in the following illustrative example.

The checking may be done in either one of two ways; that is, by multiplication or by numerical substitution.

Illustrative Example. Divide $31a - 30a^2 + 8a^3 - 15$ by $2a - 5$.

Operation

$$
\begin{array}{r|l}
 & 4a^2 - 5a + 3 \\
\hline
2a - 5 & 8a^3 - 30a^2 + 31a - 15 \\
 & \underline{8a^3 - 20a^2} \\
 & \quad -10a^2 + 31a \\
 & \quad \underline{-10a^2 + 25a} \\
 & \qquad + 6a - 15 \\
 & \qquad \underline{+ 6a - 15}
\end{array}
$$

Check. Let $a = 2$.

$$\frac{\text{Dividend}}{\text{Divisor}} = \frac{64 - 120 + 62 - 15}{4 - 5}$$

$$= \frac{-9}{-1} = 9.$$

$$\text{Quotient} = 16 - 10 + 3 = 9.$$

NOTE. In checking by numerical substitution, care must be exercised not to substitute a value which will cause the divisor to become **0**. Division by zero is meaningless in this work. Observe that $\frac{0}{5} = 0$, since $0 \times 5 = 0$. But the quotient $\frac{5}{0}$ calls for a number which, when multiplied by 0, produces 5. To avoid such contradictions, division by zero is excluded in both arithmetic and algebra.

The method to be used in dividing one polynomial by another may now be summarized as follows:

1. *Arrange the terms of both the dividend and the divisor according to the descending (or ascending) powers of the same letter.*
2. *Divide the first term of the dividend by the first term of the divisor, thus obtaining the first term of the quotient.*
3. *Multiply the divisor by the first term of the quotient and subtract this product from the dividend, thus obtaining a new dividend.*
4. *Repeat steps 2 and 3 as long as division is possible.*

This process is very similar to that of long division in arithmetic. Thus, divide 672 by 21 in the usual way. Then expand the work by dividing $(600 + 70 + 2)$ by $(20 + 1)$ by the method shown above, and compare the corresponding steps.

Exercises. *Dividing a Polynomial by a Polynomial*

Divide in each case and check the result:

1. $n^2 + 6n + 8$ by $n + 2$.
2. $d^2 + 3d + 2$ by $d + 1$.
3. $x^2 + 8x + 12$ by $x + 6$.
4. $x^2 + 7x + 12$ by $x + 4$.
5. $n^2 + 6n - 7$ by $n - 1$.
6. $b^2 + 6b - 16$ by $b - 2$.
7. $y^2 - 5y + 4$ by $y - 4$.
8. $a^2 - 3a - 10$ by $a - 5$.
9. $n^2 - 10n + 16$ by $n - 8$.
10. $w^2 - 5w - 24$ by $3 + w$.
11. $2x^2 + x - 6$ by $2x - 3$.
12. $6c^2 - 7c + 2$ by $3c - 2$.
13. $8n^2 - 22n + 15$ by $4n - 5$.
14. $20d - 8 + 12d^2$ by $8 + 4d$.
15. $6c^2 + 11cd - 10d^2$ by $3c - 2d$.

16. $4\,rs + 12\,r^2 - 5\,s^2$ by $5\,s + 6\,r$.

17. $15\,b^2 + 4\,a^2 - 23\,ab$ by $a - 5\,b$.

18. $4\,x^3 - 17\,x^2 - 16\,x + 5$ by $x - 5$.

19. $10\,p^3 + 13\,p^2 - 11\,p - 12$ by $2\,p + 3$.

20. $20\,a^3 - 15 - 22\,a^2 + 31\,a$ by $5\,a - 3$.

21. Divide $12\,x^3 - 5\,x^2 - 11\,x + 4$ by $3\,x - 2$, and check.

$$
\begin{array}{r|l}
3\,x - 2 & 4\,x^2 + x - 3 \\
\hline
12\,x^3 - 5\,x^2 - 11\,x + 4 & \\
12\,x^3 - 8\,x^2 & \\
\hline
3\,x^2 - 11\,x & \\
3\,x^2 - 2\,x & \\
\hline
-\,9\,x + 4 & \\
-\,9\,x + 6 & \\
\hline
-\,2 & \text{[Remainder]}
\end{array}
$$

NOTE. When the division is not exact we have, as in arithmetic, the relation

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{Partial Quotient} + \frac{\text{Remainder}}{\text{Divisor}}.$$

Hence the complete result for the above division is $4\,x^2 + x - 3 + \frac{-2}{3\,x-2}$.

(Why cannot the fraction $\frac{-2}{3\,x-2}$ be written immediately after the quotient without using the + sign?)

Check. Let $x = 2$. Using the above relation, we have

$$\frac{96 - 20 - 22 + 4}{6 - 2} = 16 + 2 - 3 + \frac{-2}{6-2}.$$

$$\tfrac{58}{4} = 15 + (-\tfrac{1}{2}).$$

$$14\tfrac{1}{2} = 14\tfrac{1}{2}.$$

Divide and check:

22. $3\,x^2 + 11\,x + 8$ by $x + 3$.

23. $6\,n^2 - 7\,n - 8$ by $2\,n + 1$.

24. $13\,d - 14 + 12\,d^2$ by $3\,d - 2$.

25. $10\,a^2 - 31\,ab + 15\,b^2$ by $5\,a - 3\,b$.

26. $2\,x^3 - x^2 - 11\,x + 10$ by $2\,x + 5$.

27. $10\,c^2 + 3\,c + 3\,c^3 - 4$ by $c + 2$.

28. $10\,r - 8 + 9\,r^3 - 15\,r^2$ by $3\,r - 4$.

29. $4\,n^3 - 2\,n^2p - 4\,np^2 - 16\,p^3$ by $2\,n - 4\,p$.

30. $27\,ab^2 - 45\,a^2b + 12\,a^3 + 6\,b^3$ by $4\,a - 3\,b$.

31. $2\,x^3 - 3\,x^2 - 11\,x - 3$ by $x^2 - 3\,x - 1$.

32. $6\,c^4 + 10\,c^3 + 3\,c^2 + 21\,c - 5$ by $3\,c^2 - c + 5$.

33. $19\,a - a^3 - 42\,a^2 - 2 + 10\,a^4$ by $a + 2\,a^2 - 7$.

34. $2\,n^5 + n^4 - 12\,n^3 + 7\,n^2 - 19\,n + 15$ by $2\,n^2 + 5\,n - 4$.

35. $x^3 - 19\,x + 30$ by $x + 5$.

SUGGESTION. The dividend should be written $x^3 + 0\,x^2 - 19\,x + 30$.

36. $a^3 - 8$ by $a - 2$.

37. $27\,b^3 + 1$ by $3\,b + 1$.

38. $8\,r^3 - 27\,s^3$ by $2\,r - 3\,s$.

39. $3\,x^3 - 11\,x^2 - 16$ by $x - 4$.

40. $25\,n^4 + 71\,n^2 - 12$ by $5\,n + 2$.

113. Summary. In this chapter you have learned

(1) The meaning of *base*, *exponent*, and *power*.

(2) How to combine algebraic expressions by addition and subtraction.

(3) The laws of order, grouping, exponents, and distribution for multiplication.

(4) How to multiply a monomial by a monomial, a polynomial by a monomial, and a polynomial by a polynomial.

(5) The laws of exponents and distribution for division.

(6) How to divide a monomial by a monomial, a polynomial by a monomial, and a polynomial by a polynomial.

Supplementary Honor Work

Having Fun with Exponents and Large Numbers. The following exercises will extend your knowledge of exponents. It may surprise you to discover how rapidly the powers of ordinary numbers increase as the exponents take on successively the values 1, 2, 3, and so on. Thus, study and then extend the following table of the powers of 2:

$2^1 = 2$	$2^6 = 64$	$2^{11} = 2048$	$2^{16} = ___$
$2^2 = 4$	$2^7 = 128$	$2^{12} = ___$	$2^{17} = ___$
$2^3 = 8$	$2^8 = 256$	$2^{13} = ___$	$2^{18} = ___$
$2^4 = 16$	$2^9 = 512$	$2^{14} = ___$	$2^{19} = ___$
$2^5 = 32$	$2^{10} = 1024$	$2^{15} = ___$	$2^{20} = ___$

Exercises

1. Multiply 128 by 16, using the facts given in the above table.

Solution. $128 = 2^7$

$16 = 2^4$

Product $= 2^7 \times 2^4 = 2^{11} = 2048.$ (Explain.)

2. Using the method suggested in Exercise 1, find each of the following products:

(1) 64×16. (3) 128×64. (5) 128^2.

(2) 1024×32. (4) $256 \times 8 \times 16$. (6) 51.2×6.4.

3. Divide 1024 by 128, using the above table.

Solution. $\frac{1024}{128} = \frac{2^{10}}{2^7} = 2^3 = 8.$ (Explain.)

4. Using the method suggested in Exercise 3, find each of the following quotients:

(1) $128 \div 8$. (3) $8192 \div 512$. (5) $16{,}384 \div 512$.

(2) $8192 \div 64$. (4) $4096 \div 128$. (6) $2048 \div 32$.

5. The higher powers of 2 may be obtained by a skilful use of the Law of Exponents for Multiplication.

Thus, $2^{10} = 2^5 \times 2^5 = 32 \times 32 = 1024$.

Hence $2^{20} = 2^{10} \times 2^{10} = 1024 \times 1024 = ?$

Also, $2^{25} = 2^{20} \times 2^5 = ?$

Using this method, find 2^{50}; 2^{60}; 2^{64}.

6. Everyone has had 2 parents, 4 grandparents, 8 great-grandparents, and so on. Hence we find that, going back only one generation, each person has 2 ancestors; for 2 generations he has 2^2 ancestors; for 3 generations, 2^3 ancestors; and so on.

Allowing 30 years for each generation, there have been approximately 65 generations since the beginning of the Christian Era. For this period of time, how many ancestors have you had? Is this result possible? Where is the error?

7. At 4 per cent compound interest a sum of money is doubled in about 18 years. Using this approximate value, we find that in 18 years 1 dollar amounts to 2 dollars; in 36 years, to 4, or 2^2, dollars; in 54 years, to 8, or 2^3, dollars; and so on.

If a dollar had been put at 4 per cent compound interest at the time of the founding of Montreal by Maisonneuve in 1642, it would have doubled approximately 16 times up to the present time. How much money does this represent?

8. The story is told that many years ago the inventor of the game of chess presented the game to a great Oriental monarch. When the inventor was urged to ask for a favor, as a reward for his gift, he modestly requested that he be given 1 grain of wheat for the first square on the board, 2 grains for the second, 4 for the third, and so on, the

number of grains being doubled each time up to the 64th square. Hence the wise men of the kingdom added the numbers 1, 2, 4, 8, 16, and so on. They were amazed to find that the entire harvest of the kingdom would be insufficient to meet the inventor's "modest" request.

This will not surprise you if you learn that the required number of grains was $2^{64} - 1$. Allowing 9000 grains to a pint, the required number of bushels is said to exceed the present annual wheat production of the whole world. Can you verify this statement?

9. The powers of 10 are frequently used to shorten the writing of large numbers. Thus, $10^2 = 100$; $10^3 = 1000$; $10^4 = 10{,}000$; and so on. What is the meaning of 10^9? 10^{20}? Write one trillion as a power of 10.

10. When recording measurements which are very large or very small, scientists have found it convenient to make use of exponents. For example, the extremely small particles (called molecules) which make up every substance have been measured and found to have a diameter of $\frac{1}{10^8}$ cm. What part of a centimetre does this represent? Express one billionth of a centimetre in a similar manner.

Test on Chapter VI

Find the sum and also the difference in each of the following cases:

1. $+8n$ / $-5n$
2. $-3d^2$ / $-10d^2$
3. $-12bh$ / $+20bh$
4. $+7h$ / h
5. $-21s^3$ / $+7s^3$
6. $3w - 7x + y$ / $-5w - 3x + 8y$
7. $-7n^2 + 10n$ / $n^2 - 4n - 12$

Combine the terms in each of the following cases:

8. $2x^2 - 3 + 5x^2 - x + 9 - 3x - 4x^2$.

9. $-10ab^2 + 2a^2b - 3a^2b + 5ab^2 + 7a^2b$.

In each of the following exercises select the correct answer from those given in the parenthesis:

10. $(-3w) \times (-6w) = (-18w, -9w^2, +18w^2)$.

11. $(-12d^3) \times d^2 = (-12d^5, -11d^5, -12d^6)$.

12. $6ab^2 \times 24a^2 = (30a^3b^2, 144a^2b^2, 144a^3b^2)$.

13. $(-28a^2b^5) \div (-4a^2b) = (-7a^2b^4, 7ab^4, 7b^4)$.

14. $d(1 - 10d) = (d - 10d^2, 1 - 10d^2, d - 10d)$.

15. $-2n^2(n^3 - 4n + 6) = (-2n^6 + 8n^2 - 12n^2, -2n^5 + 8n^3 - 12n^2, 2n^5 - 8n^3 - 12n^2)$.

16. $\dfrac{4r^3 - 8r^2 + 10r}{-2r} = (4r^3 - 8r^2 - 5, -2r^2 - 8r^2 + 10r, -2r^2 + 4r - 5)$.

Remove the parentheses and combine the resulting terms in Exercises 17–19:

17. $7 - 2(5a + 3)$.

18. $3y + (4x - 2y)$.

19. $4n(1 - n) + (n^2 + 3) - (2n - 1)$.

20. Multiply $2y^2 - 7y + 3$ by $y + 5$.

21. Divide $6c^3 - 5c^2 + 10c - 3$ by $3c - 1$.

22. The dimensions of a rectangle are w and $3w - 8$. A second rectangle has the dimensions $w + 4$ and $4w - 10$.

(1) Find the sum of the perimeters of these rectangles.

(2) Find the sum of the areas of these rectangles.

(3) By how much does the area of the second rectangle exceed that of the first?

Exercises. *Review*

1. Solve each of the following equations and check the result:

(1) $3x = 18$.
(2) $n + 9 = 19$.
(3) $2c - 5 = 15$.
(4) $\frac{a}{4} = 8$.
(5) $\frac{2a}{3} = 12$.
(6) $7p = 3p + 12$.
(7) $5n - 7 = n + 13$.
(8) $4d + 9 = 2d + 10$.
(9) $6s - 12 + s = 2s + 18$.
(10) $3b + 5 - b = 10 - 2b + 7$.

2. The formula $v = u + at$ is commonly used in physics.
(1) Find the value of u when $v = 45$, $a = 5$, and $t = 7$.
(2) Find the value of a when $v = 30$, $u = 3$, and $t = 10$.

3. The formula $g = .03264\,td^2$ is used in computing the discharge in gallons from a pump. Find g if $t = 80$ and $d = 18$.

4. A salesman's regular monthly salary, in dollars, is s. He also receives a commission of 3 dollars for each sale. Write a formula for his monthly income i, if the number of sales is represented by n.

5. The following table shows the cost of mailing parcels of certain weights to an adjoining province.

Weight in pounds (p) =	1	2	4	7	10
Postage in cents (c) =	10	17	31	52	73

(1) Write a formula showing the relation of c and p.
(2) Using this formula, find c if $p = 3$; if $p = 9$.

6. Can you find three numbers whose sum is 34, such that the first two are consecutive and the third equals the sum of the first two?

7. Write a formula suggested by the following table:

If $x =$	1	3	6	8	12	15	20
then $y =$	8	14	23	29	41	50	65

8. The formula for the distance d which a person can travel in any number of hours h, if he travels at the rate of 16 miles an hour, is $d = 16\,h$.

(1) Draw a graph of this formula.

(2) From the graph find the distance covered in $5\frac{1}{2}$ hours.

(3) From the graph determine how many hours it will take to cover a distance of 128 miles.

9. Draw the graph of the formula $y = 4\,x$, using in succession the values $x = 0, 1, 2, 3, 4, 5, 6, 7$.

10. Draw the graph of the formula $y = 2\,x + 1$, using in succession the values $x = 0, 1, 2, 3, 4, 5, 6, 7$.

Multiply in each case and check the result:

11. $(9\,y - 2)(3\,y + 5)$.
12. $(4\,x + 1)(3\,x - 7)$.
13. $(xy - 6)(3\,xy + 7)$.
14. $(4\,a - 7\,b)(3\,a - 9\,b)$.
15. $(y - 7 + 5\,y^2)(3\,y - 2)$.
16. $(5\,x^3 - 2\,x + 4)(3\,x^2 - 4)$.
17. $(a^3 - 4\,a^2 + 5\,a - 2)(a^2 + 5\,a)$.
18. $(5\,n^3 + 9 - 2\,n^2 + 6\,n)(n - 3)$.
19. $(x^3 - 2\,x^2 + 7\,x - 3)(2\,x^2 + 4\,x - 5)$.
20. $(a^3 - 3\,a^2b + 4\,ab^2 - b^3)(a^2 - 2\,ab + b^2)$.

21. If $P = 2\,n^2 + 7\,n - 15$, $Q = 10\,n^2 - 29\,n + 21$, and $R = 2\,n - 3$, find

(1) $2\,P + 3\,Q - 4\,R$.
(2) $(P + Q) \div R$.
(3) $P \times Q$.
(4) $(P \div R) + (Q \div R)$.

CHAPTER VII

EQUATIONS OF THE FIRST DEGREE IN ONE UNKNOWN

114. Equations of the First Degree. An *equation of the first degree in one unknown* is one of such a form that

(1) the unknown appears only with the exponent 1;
(2) the unknown does not appear in any denominator.

For example, $3x + 7 = 9$, $15 = 7 - 2x$, and $4x + 5 = 9 + x$ are equations of the first degree.

Such equations are sometimes called *simple equations*, or *linear equations*. The latter name is used because the graph of such an equation is always a straight line.

An equation such as $4x^2 + 7x = 5$ is said to be of the *second degree*, since the term $4x^2$ contains the *second* power of x.

115. Short Methods of Solving. Only very simple equations of the first degree have been introduced thus far. The ready solution of more complicated equations of the first degree does not require any new principle. It does involve, however, a more *extended use* of these principles. Some of the most convenient special methods for solving such equations will be considered in this chapter.

116. Interchanging Members. It is evident that the two members of an equation may be interchanged without changing the value of the unknown number.

For example, the equation $7 = 5x - 3$ may be written in the more convenient form $5x - 3 = 7$.

Examine each of the following equations and observe how the solution is simplified by interchanging the members.

Original Form	New Form
(1) $9 = a + 7.$	$a + 7 = 9.$
(2) $20 = 4a - 4.$	$4a - 4 = 20.$
(3) $8 = n - 7.$	$n - 7 = 8.$
(4) $100 = 20x + 80.$	$20x + 80 = 100.$
(5) $9 = 5x - 1.$	$5x - 1 = 9.$

117. Transforming an Equation by Addition or Subtraction. By a skilful use of the axioms of addition and subtraction a complicated equation can often be changed, or *transformed*, into an equation of a simpler type. How this is possible will become apparent from a study of the following statements:

(1) $a + 5 = b$	(1) $a - 5 = b$
$5 = 5$	$5 = 5$
(2) $a = b - 5$	(2) $a = b + 5$

That is, if $a + 5 = b$, we have $a = b - 5$, by Axiom 2.

Also, if $a - 5 = b$, we have $a = b + 5$, by Axiom 1.

In the *first* case, the $+ 5$ of the first member disappears by subtraction, and $- 5$ appears in the second member. In the *second* case, the $- 5$ of the first member disappears by addition, and $+ 5$ appears in the second member.

Similarly, $7x - 5 = 9$ becomes $7x = 9 + 5$, or $7x = 14$.

Also, $9x = 4 + 7x$ becomes $9x - 7x = 4$, or $2x = 4$.

This short method of applying these axioms may be stated in the form of a *general rule*, as follows:

A term may be dropped from either member of an equation, provided the corresponding opposite term is written in the other member.

That is, if $+7$ is dropped from the first member, we write -7 in the second member; if $-2x$ is dropped from the second member, we write $+2x$ in the first member; and so on.

In applying this rule, it *looks as if* the original term had been *moved* from one member of the equation to the other, with an opposite sign. Hence this rule is often called the *rule of transposition*, and is stated in the following form: *A term may be transposed from one member of an equation to the other, provided its sign is changed.*

The rule stated above will be called the *Rule of Transformation by Addition or Subtraction*, and the abbreviation Tr. may be used as a short way of referring to it.

Illustrative Example 1. Solve the equation

$$9n - 2 + 5n = 14n + 6 - 4n.$$

Solution. The equation may be transformed so that we obtain in the first member only terms containing n and in the second member only numerical terms.

Hence we have $9n + 5n - 14n + 4n = 6 + 2.$

By the Rule of Similar Terms, we have

$$4n = 8.$$

Hence $$n = 2.$$

Check. If 2 is substituted for n in the original equation, show that you obtain an identity.

Illustrative Example 2. Solve the equation

$$15x - 10 - 3x = 7x + 26 - x.$$

Solution. Using the method explained above, we have

(1)	$15x - 3x - 7x + x = 26 + 10.$	Tr.
(2)	$6x = 36.$	S. T.
(3)	$x = 6.$	D_6.

Check. Substitute 6 for x in the original equation.

NOTE. After the student has mastered this short method of solving equations, much of the work shown above may be performed mentally. For example, the similar terms in each member may be combined before the equation is transformed.

Exercises. *Applying the Rule of Transformation*

Numbers 1 to 10, oral

Solve each of the following equations by the method explained above. Interchange the members whenever it is convenient.

1. $4a - 2 = 3a$.
2. $7n - 10 = 5n$.
3. $2n + 14 = 7n + 9$.
4. $5x = 12 + x$.
5. $9a - 4 = 5a + 8$.
6. $8c + 9 = 5c - 6$.
7. $5x - 7 = 9 - 3x$.
8. $6a = 18 - 3a$.
9. $4p - 3 = 3p + 4$.
10. $d + 7 = 31 - 5d$.
11. $7w + 1 = w + 25$.
12. $12y - 5 = 34 - y$.
13. $5x - 2 + 3x = x + 12$.
14. $n + 3 + 2n = 23 - 2n$.
15. $8d - 1 = 6d + 1 - 2d$.
16. $3p - 8 + 4p = p + 28$.
17. $9w + 4 - 2w = 3w + 6$.
18. $7x + 15 - x - 3 = 36$.
19. $11s = 8s + 30 - 3s - 6$.
20. $12r - 15 = 9r + 9 - 3r$.
21. $6y - 3 + 2y = y + 7 + 2y$.
22. $10n + 6 - n = 31 - n$.
23. $4a - 3 = a + 18$.
24. $5t - 1.07 = 2.5 - 2t$.
25. $.7n - 5 = .4n + 7$.
26. $.06t - 12 = .04t + 18$.
27. $1.2y + 1.4 = 5.6 - .9y$.
28. $3.7d + 7 = 2.9d + 9.2 - 1.2d$.

118. Equations with Negative Roots. In solving equations involving negative roots, we shall likewise use only the principles that have been considered in the preceding

chapters. This will become clear from a study of the following example:

Illustrative Example. Solve the equation

$$5x - 2 - 2x = x - 10.$$

Solution.

(1)	$5x - 2 - 2x = x - 10.$	Given.
(2)	$3x - 2 = x - 10.$	S. T.
(3)	$3x - x = -10 + 2.$	Tr.
(4)	$2x = -8.$	S. T.
(5)	$x = -4.$	D_2.

Check. $5(-4) - 2 - 2(-4) = (-4) - 10.$

$$-20 - 2 + 8 = -4 - 10.$$

$$-14 = -14.$$

119. Changing Signs in an Equation. It is sometimes convenient to transform an equation by multiplying each member by -1. This process changes the sign of each term appearing in the equation.

Examine each of the following equations and observe how the solution is simplified by multiplying each member by -1.

	Original Form	New Form
(1)	$-4x = 8.$	$4x = -8.$
(2)	$2 - 3x = -10.$	$-2 + 3x = 10.$
(3)	$-5x + 6 = -2x.$	$5x - 6 = 2x.$
(4)	$-7 - 5x = 14 - 2x.$	$7 + 5x = -14 + 2x.$

Exercises. *Finding Negative Roots*

Solve each of the following equations and check each result:

1. $x + 7 = 5.$
2. $12 = 18 + n.$
3. $2n + 17 = 9.$
4. $15 = 12 - p.$
5. $5d - 3 = -28.$
6. $x = 8x + 21.$

7. $3w - 10 = 5w$.
8. $12b = 7b - 25$.
9. $9y + 12 = y - 12$.
10. $7r - 6 = 3r - 30$.
11. $4a + 3 = 21 + 10a$.
12. $2t - 30 = 5t - 15$.
13. $s + 28 = 13 - 4s$.
14. $-4n + 9 = 2n + 12$.
15. $10 - 3d = d + 13$.
16. $6x - 3 + 2x = 9x + 5$.
17. $7g - 4 = 13g + 5 - 3g$.
18. $6s + 7 - s = 8s + 11$.
19. $4x - 3.2 + x = 2x - 10.4$.
20. $7 - 3d - 2.7 = d - 8.1 - 8d$.

21. The sum of two numbers is -39, and one number is 15 more than the other. Find the numbers.

22. The sum of three numbers is -8. The second is 5 more than the first, and the third is 1 less than twice the first. What are the numbers?

23. The length of a rectangle is 15 inches more than twice its width. If the perimeter is 24 inches, what are its dimensions? Is this result possible?

24. The total profit from a certain business, for 2 years, was \$2200. During the second year the profit was \$500 more than during the first year. What was the profit of each year? Interpret your answers.

25. A real-estate agent sold three houses. His profit from the second was \$1300 more than that from the first, and the profit from the third was \$400 less than twice what he made on the second. If his total profit was \$1500, how much did he make on each sale?

120. Equations involving Parentheses. When more than one unknown quantity appears in the same problem, it is often necessary to use parentheses in forming the equation that represents the conditions of the problem. It is important, therefore, to develop skill in solving such equations.

Illustrative Example. The sum of two numbers is 29. Four times the larger number exceeds five times the smaller number by 8. Find the numbers.

Solution. Let n = the larger number.

Then $29 - n$ = the smaller number.

But $4 \times$ (larger number) $- 5 \times$ (smaller number) $= 8$.

Hence we have the following equation and its solution:

	(1)	$4n - 5(29 - n) = 8.$	Given.
	(2)	$4n - 145 + 5n = 8.$	Why?
	(3)	$9n - 145 = 8.$	S. T.
	(4)	$9n = 153.$	A_{145}.
	(5)	$n = 17.$	D_9.
Hence	(6)	$29 - n = 12.$	Why?

That is, the required numbers are 17 and 12.

Check. $4 \times 17 - 5 \times 12 = 8.$

That is, $68 - 60 = 8,$

or $8 = 8.$

Exercises. *Equations containing Parentheses*

Solve each of the following equations and check each result:

1. $3(x + 2) = 12.$
2. $7(x - 5) = 21.$
3. $4(3x + 7) = 4.$
4. $5(2x - 9) = 15.$
5. $8(3a - 2) + 5 = 13.$
6. $12(a + 5) - 15 = -3.$
7. $9(5a - 8) - 20a = 3.$
8. $9n + 33 = 6(4n + 3).$
9. $7(a + 12) = 14 - 3a.$
10. $15(2n - 1) + 8 = 5n - 57.$
11. $2(4n + 11) = 3(5n - 2).$
12. $9x - 2(x + 5) = 11.$
13. $12 + 3(2x - 1) = 6.$
14. $5n + (4n + 6) = 42.$
15. $7y - (4y - 1) = 28.$
16. $9n - (3n + 5) = n - 50.$
17. $5(x - 2) = 4(2x - 4).$
18. $3(2s + 5) - 2(5s - 1) = 25.$

19. $3(d + 9) = 4(5 - 3\,d) + 22.$

20. $8(3\,n + 1) = 21 + (8\,n - 5).$

21. $10\,n - 3 = (6\,n - 11) - (n + 7).$

22. $12\,a - (9 - 3\,a) = 4(6\,a - 9).$

23. $3\,x + x(x - 5) = 8 + (2 + x^2).$

24. $4\,y^2 - 2\,y(2\,y + 7) = 4(15 - y).$

25. $2(b^2 - 3) - b(5 + 2\,b) + 26 = 0.$

26. $h(4\,h - 1) - (3\,h - 3) = 4\,h^2.$

27. $x(3\,x - 1) - (5\,x + 3) = 3(x^2 - 2).$

28. $6(d - 3) - 5(4 + 3\,d) + 10 = 2(3\,d + 16).$

29. $6\,n - 2(4 - n) + 12 = 10 - (4\,n + 12).$

30. $3(7\,w - 1) - 25 = 4(w + 3) + (3\,w - 5) - 7.$

31. A line 20 inches long is to be divided into two parts such that one part is 8 inches more than the other. Find the two parts.

32. A board 16 feet long is to be sawed into two pieces one of which is to be 3 feet longer than the other. Find the length of each piece.

33. A rectangular building lot has a perimeter of 282 feet. Its length is 5 feet more than three times its width. What are its dimensions?

121. Equations with Fractional Coefficients. Many problems lead to equations which involve fractional coefficients. An equation of this sort may easily be transformed into one which contains only whole numbers, by multiplying both members of the equation by a number which contains each

of the denominators as one of its factors. The resulting equation can then be solved in the usual way.

Illustrative Example 1. Solve the equation

$$\frac{x}{3}+2=\frac{1}{3}.$$

*Solution.** Multiplying both members by 3, we have

$$3\left(\frac{x}{3}+2\right)=3\times\frac{1}{3}.$$

By the Rule of Distribution, we have

$$x+6=1.$$

Hence, by Axiom 2, $x=-5$.

Check. Substituting -5 for x in the given equation, we have

$$\frac{-5}{3}+2=\frac{1}{3},$$

or

$$\tfrac{1}{3}=\tfrac{1}{3}.$$

Illustrative Example 2. Solve the equation

$$\frac{3n}{4}-\frac{n}{8}=\frac{5}{2}.$$

Solution.

(1)	$8\left(\frac{3n}{4}-\frac{n}{8}\right)=8\times\frac{5}{2}.$	M_8.
(2)	$6n-n=20.$	Why?
(3)	$5n=20.$	S. T.
(4)	$n=4.$	D_5.

Check.

$$\frac{3\times 4}{4}-\frac{4}{8}=\frac{5}{2},$$

or

$$3-\tfrac{1}{2}=\tfrac{5}{2},$$

or

$$\tfrac{5}{2}=\tfrac{5}{2}.$$

In the exercises which follow, only simple fractions will be used. Equations of a more complicated type will be considered at a later stage.

* You will recall that when a fraction is multiplied by a whole number, its numerator only is multiplied by that number. Thus,

$4\times\frac{3}{5}=\frac{12}{5}$, $3\times\frac{5}{7}=\frac{15}{7}$, and so on.

Similarly, in algebra, $3\times\frac{x}{5}=\frac{3x}{5}$, $5\times\frac{3x}{4}=\frac{15x}{4}$, and so on.

Exercises. *Equations with Fractional Coefficients*

Solve each of the following equations and check each solution:

1. $\frac{x}{2}+3=5$.
2. $\frac{x}{5}+\frac{x}{10}=3$.
3. $\frac{n}{6}+4=\frac{n}{3}$.
4. $\frac{d}{2}-\frac{d}{6}=\frac{2}{3}$.
5. $\frac{3y}{4}-5=y$.
6. $\frac{2b}{5}-\frac{b}{15}=\frac{5}{3}$.
7. $\frac{2n}{5}+\frac{7}{20}=\frac{3n}{4}$.
8. $\frac{5n}{3}-\frac{1}{4}=\frac{5n}{12}+1$.
9. $\frac{c}{2}-\frac{5c}{8}+\frac{3c}{4}=5$.
10. $a-\frac{a}{3}=\frac{12}{9}$.
11. $\frac{d}{3}+4=\frac{d}{6}$.
12. $\frac{y}{8}-\frac{2y}{3}=\frac{13}{24}$.
13. $x-\frac{x}{5}=20$.
14. $n+\frac{n}{20}+\frac{n}{4}=52$.
15. $\frac{b}{6}-4=\frac{5b}{18}$.
16. $\frac{3r}{4}=\frac{5r}{8}-4$.
17. $\frac{n+5}{12}=\frac{3}{4}$.
18. $\frac{s+4}{4}=\frac{s}{8}$.
19. $\frac{n}{3}+\frac{n}{5}+14=n$.
20. $\frac{t-3}{6}=\frac{t}{5}+2$.

21. $.3x+.2=2.3$.

SUGGESTION. Multiply both members of the equation by 10, which will remove the decimals.

22. $.25n=.13n+24$.
23. $.35x+2.1=.14x$.
24. $.15y=.25(y-2)$.
25. $.75(n+20)=n-10$.

26. The difference between $\frac{3}{4}$ of a number and $\frac{1}{8}$ of the number is 35. Find the number.

27. The width of a building lot is $\frac{3}{8}$ of its length. Find its dimensions if the perimeter is 330 feet.

28. A farmer decided to sell $\frac{1}{3}$ of his apple crop, to keep $\frac{1}{4}$ of it, and to put the remainder in cold storage. If he put 50 bushels in cold storage, how many bushels were there in his entire crop?

29. A coat which had been marked down $\frac{1}{5}$ sold for \$24.60. What was the original price?

30. There are two numbers such that the larger is 6 more than the smaller. One half of the larger is 7 more than one sixth of the smaller. Find the numbers.

122. Literal Equations. A *literal equation* is one in which at least one of the coefficients or terms appearing in the equation is represented by a letter, aside from the letter representing the unknown quantity. For example, $x+a=b$, $3x+c=d-5$, and $x-3=p$ are literal equations in which x represents the unknown quantity. Such an equation may be solved by applying the rules stated previously.

Sometimes it becomes necessary to find one of the letters of a *formula* in terms of the other letters. Simple illustrations of such transformations were suggested in a previous chapter. If a formula is more complicated, it is best to regard it as a literal equation. The four fundamental axioms or the special methods explained previously may then be used to express any letter in terms of the other letters, as shown in the following examples:

Using Addition	Using Subtraction	Using Multiplication	Using Division
$x-b=c$	$x+b=c$	$\frac{x}{a}=b,$	$ax=b,$
$b=b$	$b=b$		
$x=c+b$	$x=c-b$	$a\cdot\frac{x}{a}=a\cdot b,$	$\frac{ax}{a}=\frac{b}{a},$
		$x=ab.$	$x=\frac{b}{a}.$

Illustrative Example 1. If $2x - 4b = 10c + 2b$, find x.

Solution.

(1)	$2x = 10c + 6b.$	Tr. and S. T.
(2)	$x = 5c + 3b.$	D_2.

Illustrative Example 2. If $ab + c = d$, find a.

Solution.

(1)	$ab = d - c.$	S_c.
(2)	$a = \frac{d-c}{b}.$	D_b.

Exercises. *Literal Equations*

Numbers 1 to 14, oral

Solve the following equations for x or y:

1. $2x = n.$
2. $nx = 4.$
3. $ay = b.$
4. $x + 3 = n.$
5. $y + n = 8.$
6. $y + d = c.$
7. $x - 5 = a.$
8. $x - n = 4.$
9. $y - r = s.$
10. $2x + 3n = 9n.$
11. $3y - a = b.$
12. $\frac{x}{3} = c.$
13. $\frac{y}{d} = 6.$
14. $\frac{x}{n} = a.$

Solve the following equations for x or y, and check each result:

15. $2x + 6 = 4n.$
16. $5y - 6a^2 = 4a^2.$
17. $3x - 12c = 3d.$
18. $7a = 4x + b.$
19. $2y - 5d = 10d - 3y.$
20. $c + 3y = 2y + d.$
21. $7x - 3m = 4x + 6n.$
22. $3(x - a) = x + 7a.$
23. $8s - 2(y + 3s) = 12s.$
24. $2(5y - 3a) = 4(y - 3b).$
25. $ax - ab = a^2.$
26. $nx + c = 2r.$
27. $3b^2y = 2b(6b^2 + by).$
28. $4c(5c - 2x) = 3(4c^2 - 2cx).$

29. $\frac{y}{2} - \frac{h}{3} = \frac{h}{6}$.

30. $\frac{x}{a} + a = 3$.

31. $\frac{x-b}{3} = 2d$.

32. $2 + \frac{2x}{10r} = \frac{x}{3r}$.

The formulas below are literal equations which may be solved for any one of the letters used in terms of the remaining letters.

33. Solve $A = lw$ for w; for l.

34. Solve $P = W + F$ for W; for F.

35. Solve $p = s - c$ for s; for c.

36. Solve $D = \frac{M}{V}$ for M; for V.

37. Solve $A = \frac{bh}{2}$ for h; for b.

38. Solve $v = u + at$ for u; for t; for a.

39. Solve $A = \frac{u+v}{2}$ for u; for v.

40. Solve $F = \frac{Wa}{g}$ for a; for g; for W.

41. Solve $w = \frac{2bd^2}{3l}$ for b; for l.

42. Solve $l = a + d(n-1)$ for a; for n.

43. Solve $A = p + prt$ for t; for r.

44. Solve $S = 2\pi rh$ for h; for r.

45. Solve $T = \frac{l}{a} + t$ for l; for t.

123. Percentage Problems. By using the formulas $p = br$ and $i = prt$, as well as the short methods of solving equations which have been considered in this chapter, you will now be able to solve more difficult applied problems involving percentage. In the following exercises use the

suggestions which have been given previously for solving such problems. In the case of interest problems, only simple interest is to be computed.

Exercises. *Percentage Problems*

1. An agent sells a house for \$9350, making a profit of 10 per cent. What was the original value of the house?

2. A school building cost \$367,500, which exceeded the original estimate by 5 per cent. What was the estimated cost?

3. Two equal sums of money are invested, one at 4 per cent and the other at 5 per cent. If the total interest from both investments is \$270 a year, how large is each sum?

4. Two sums of money are invested at 4 per cent and 6 per cent respectively. The second is \$2000 larger than the first. If the combined annual income from both investments is \$360, what are the two sums?

SUGGESTION. The conditions of the problem may be tabulated as follows:

	PRINCIPAL	RATE PER CENT	INTEREST
First sum . .	p	4%	$.04\,p$
Second sum. .	$p + 2000$	6%	$.06(p + 2000)$

Hence the required equation is

$$.04\,p + .06(p + 2000) = 360.$$

5. Two sums of money, the first exceeding the second by \$1500, were invested at 4 per cent and 5 per cent respectively. The larger sum yielded \$15 more each year than the other. How large were the two sums?

6. How many dollars must be invested at 4 per cent to yield an annual income of $3000?

7. A man invests $4000, a part at 4 per cent and the remainder at 6 per cent. If the 4 per cent investment yields annually $60 more than the 6 per cent investment, how many dollars were invested at each rate?

8. A man invests $2000 at 4 per cent, $3000 at 5 per cent, and $1200 at 6 per cent. What rate of interest is he receiving from the three investments combined?

9. At what rate is a sum of money invested if the interest each year equals one tenth of the principal? one twentieth of the principal?

SUGGESTION. In this case $\frac{r}{100} \cdot p = \frac{1}{10} \cdot p.$

Dividing by p, we have $\frac{r}{100} = \frac{1}{10}.$

Hence $r = 10.$

That is, the rate is 10%.

10. In how many years will the interest on a principal invested at 4 per cent simple interest equal the principal?

SUGGESTION. Since $\frac{4}{100} \cdot p \cdot t = p,$

we have $\frac{4}{100} t = 1;$

whence $t = 25.$

11. What is the result, in Exercise 10, if the rate is 5 per cent? $4\frac{1}{2}$ per cent? $3\frac{1}{2}$ per cent?

12. In how many years does a sum invested at 5 per cent simple interest become three times as large?

13. In measuring a line which was 25 inches long a pupil made a mistake of $\frac{1}{4}$ inch. What was the relative error? the percentage error?

EXPLANATION. All measurements are only approximately correct, since they always involve an error. By dividing the *actual error* of a

measurement by the numerical value under consideration, we obtain the *relative error*. If the relative error is expressed in per cent form, we obtain the *percentage error*. For example, suppose that in measuring a line which is 25 inches long you make an error of $\frac{1}{2}$ inch. Then the *relative error* is $\frac{\frac{1}{2}}{25}$, or $\frac{1}{50}$, and the *percentage error* is 2%.

14. In measuring a line that was 4 feet long a pupil made a mistake of 5 per cent. What was the actual error, in inches?

15. The length of a playground was 250 feet. It was measured by a pupil, with a percentage error of $\frac{1}{2}$ of 1 per cent. What was the actual error, in inches?

16. A student sold books on a commission of 12 per cent. His commission in one week amounted to \$84. Find the amount of his sales.

17. The population of Winnipeg in a recent year was 220,000. This represents an increase of 22 per cent in 10 years. What was the population 10 years previous?

18. Suppose that on a debt of \$5000 the interest rate is increased each year by 1 per cent. If after 3 years this debt amounts to \$5750, what is the rate of interest for each year?

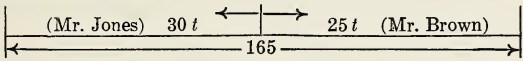
124. Motion Problems. The formula $d = rt$, which was introduced in a previous chapter, enables us to solve many interesting problems that involve a uniform rate of motion.

Illustrative Example 1. Two neighbors, Mr. Brown and Mr. Jones, start on an automobile trip at the same time, one going east and the other going west. Mr. Brown travels at the rate of 25 miles per hour, and Mr. Jones at the rate of 30 miles per hour. After how many hours will they be 165 miles apart?

Solution. Let t represent the number of hours. Then the conditions of the problem may be tabulated as follows:

	Rate (r)	Time (t)	Distance (d)
Mr. Brown	25	t	$25\,t$
Mr. Jones	30	t	$30\,t$

A diagram such as the following may also be used to picture the conditions of the problem:



Hence, since the sum of the two distances equals 165, we have

$$25\,t + 30\,t = 165.$$

Solving the equation, we find that $t = 3$.

That is, in 3 hours the two men will be 165 miles apart.

Check. Show that $3 \times 25 + 3 \times 30 = 165$.

Illustrative Example 2. Henry started from camp one morning at 6 A.M. to get some supplies in the nearest town. He walked at the rate of 3 miles an hour. At 8 A.M. his friend Jack followed him on his bicycle, going at the rate of 9 miles an hour. At what time, and how far from camp, did he overtake Henry?

Solution. The conditions of the problem may be tabulated as follows:

	Rate	Time	Distance
Henry	3	t	$3\,t$
Jack	9	$t - 2$	$9(t - 2)$

Since Jack overtakes Henry, the two distances are equal.

Hence $$9(t - 2) = 3\,t.$$

Solving this equation, we find that

$$t = 3.$$

(Can you complete the solution?)

Exercises. *Motion Problems*

1. Two trains start from the same station at the same time, one going north and the other going south. Their hourly rates are 40 and 50 miles respectively. After how many hours will the trains be 360 miles apart?

2. Two trains start from the same place at the same time, travelling in opposite directions. One travels twice as fast as the other. If they are 360 miles apart in 4 hours, find the speed of each train.

3. A starts from a certain place and walks at the rate of 4 miles per hour. Two hours later B starts from the same place and travels in the same direction at the rate of 6 miles per hour. In how many hours will B overtake A?

4. In Exercise 3, how many hours after B started will his distance exceed that of A by 10 miles?

5. In Exercise 3, how many hours after A started will he be only 2 miles ahead of B?

6. Two trains start east over the same road, at the same time, one from Regina and the other from a point 30 miles west of Regina. If the first train runs at a rate of 40 miles per hour and the second at a rate of 50 miles per hour, in how many hours will the second train overtake the first?

7. In the preceding problem, in how many hours will the faster train be only 5 miles behind the other?

8. A messenger is sent from a camp at 7 A.M. to the nearest post office. Another messenger follows him at 8 A.M. and overtakes him after 1 hour and 30 minutes. If the second messenger travels 4 miles an hour faster than the first, what is the rate of each?

9. A freight boat which is averaging 10 miles per hour is 900 miles out at sea when it is followed by a fast passenger boat. If the passenger boat makes 25 miles an hour, in how many days will it overtake the freight boat?

10. A passenger train averaging 38 miles an hour leaves one terminal station of a railroad at 5 P.M. At the same time a freight train running at a rate of 22 miles per hour leaves the other terminal station. If the two stations are 380 miles apart, at what time will the trains pass each other?

11. A sailboat starts out from a harbor, going at the rate of 8 miles per hour. It is followed after 2 hours by a motor boat which is going at the rate of 16 miles per hour. How long will it take the motor boat to overtake the sailboat?

12. In an automobile race one car makes 60 miles per hour. A second goes at an average speed of 54 miles per hour. If the second car has started 20 minutes before the other car, how soon will it be overtaken by the faster car?

13. A freight train which averages 22 miles per hour leaves Ottawa for Winnipeg at 6 P.M. It is followed on the same road at 9 P.M. by a fast passenger train. In how many hours will the fast train overtake the first train if the fast train averages 40 miles per hour?

14. A steamship which is in distress sends out an S O S signal by wireless. The ship is then 390 miles from the nearest harbor and is able to make only 6 miles per hour. How soon will a relief ship sent from the harbor at the rate of 24 miles per hour be able to reach the disabled ship?

15. The distance from New York to Liverpool, on the route ordinarily chosen, is about 3105 miles. A passenger

boat leaves New York for Liverpool at 9 A.M. on Saturday, and averages 22 miles per hour. A second boat leaves Liverpool for New York at 9 A.M. on the following Tuesday, Eastern Standard Time. When will the boats meet if the second boat makes 23 miles per hour?

125. Mixture Problems. In the grocery business it is often necessary to mix various qualities of groceries in order to obtain a "blend" that will sell at a reasonable price. In like manner, "mixture problems" must be solved by druggists, jewellers, chemists, manufacturers, and so on. A few of the simplest types of these problems will be introduced in the following exercises.

Illustrative Example. A wholesale grocery firm wishes to mix two grades of coffee, worth 40 and 60 cents a pound, so as to obtain a grade that can be sold at 42 cents a pound. How many pounds of each kind must be used to obtain 90 pounds of the mixture?

Solution. Let n = the number of pounds of 40-cent coffee.

Then $90 - n$ = the number of pounds of 60-cent coffee.

Hence $40\,n$ = the value of the 40-cent coffee, in cents,

and $60(90 - n)$ = the value of the 60-cent coffee, in cents.

But 90×42 = the value of the mixture, in cents.

Hence $40\,n + 60(90 - n) = 3780$.

Solving this equation, we find that

$$n = 81.$$

(Can you complete the solution?)

Exercises. *Mixture Problems*

1. A wholesale house wishes to blend 200 pounds of 55-cent coffee with 300 pounds of a cheaper grade, and thus be able to sell the new blend for 46 cents a pound. What grade of coffee must be used for blending?

2. A grocer decides to mix almonds which sell for 25 cents a pound with 120 pounds of brazil nuts which sell for 38 cents a pound. How many pounds of almonds must he use if he intends to sell the mixed nuts for 30 cents a pound?

3. A confectioner wishes to mix candy which sells for 30 cents a pound with a grade that sells for 50 cents a pound, so that he can sell the mixture for 38 cents a pound. If he needs 50 pounds of this mixture, how many pounds of each kind should he use?

4. A grocer needs 30 pounds of mixed spices which are to sell at 35 cents a pound. How many pounds of spices which sell for 25 cents a pound and for 40 cents a pound, respectively, must he use to get the desired mixture?

5. How much alcohol must be added to 8 ounces of a 25 per cent solution of iodine to reduce it to a 10 per cent solution?

SUGGESTION. If n ounces of alcohol are added, the mixture will contain $(n + 8)$ ounces. Hence $\frac{10}{100}(n + 8) = \frac{25}{100} \times 8$. Multiply each member by 100.

6. How much water must be added to 3 ounces of a 15 per cent solution of argyrol to reduce it to a 10 per cent solution?

7. Camphor gum is dissolved in alcohol to make spirits of camphor. How many ounces of alcohol must be added to 24 ounces of a 15 per cent camphor solution to make a 10 per cent solution?

8. How much pure silver must be added to 120 ounces of an alloy containing 75 per cent of silver, so that the resulting alloy shall be 90 per cent pure?

SUGGESTION. If n ounces of silver are added, we have

$$\frac{90}{100}(n + 120) = \frac{75}{100} \times 120 + n. \quad \text{(Why?)}$$

9. An auto radiator contains 8 quarts of a 10 per cent solution of alcohol. How much alcohol must be added to make it a 30 per cent solution?

10. A school ordered 300 pounds of nitric acid 67 per cent pure. How much water had to be added to reduce it for use in the laboratory, which required only a 30 per cent solution?

11. Sulphuric acid 98 per cent pure is purchased by a school for use in its chemical laboratory. How much water must be added to 150 pounds to reduce it to a 15 per cent solution, the strength desired for laboratory use?

12. A firm ordered 3675 pounds of sulphuric acid 80 per cent pure. The manufacturer shipped a smaller quantity of a solution 98 per cent pure, in order to save on transportation. How much water had to be added to the shipment to obtain the desired quantity of an 80 per cent solution?

13. How much water must be evaporated from 30 gallons of a 15 per cent salt solution so that the remainder will be a 25 per cent solution?

14. How many quarts of cream containing 25 per cent butter fat and of milk containing 5 per cent butter fat must be mixed so as to obtain 30 gallons of cream which will contain 20 per cent butter fat?

Exercises. *Miscellaneous Problems*

1. When 5 is subtracted from one half of a number, the result is 6. Find the number.

2. The sum of one half of a number and one third of the number is 10. Find the number.

3. If one fifth of a number is subtracted from 55, the result is twice the number. Find the number.

4. Louise has 54 cents, and Helen has 32 cents. How many cents does Louise have to give to Helen so that each may have the same number of cents?

5. James has a dollars, while Robert has only b dollars. How many dollars does James have to give to Robert so that each may have the same number of dollars?

6. A father divides $63 among his three sons and two daughters, giving to each son $1 more than to a daughter. How many dollars does he give to each son and daughter?

7. In the assembly hall of a school it was found that when 30 pupils were seated in a row, only one seat was occupied in the last row. If only 29 pupils had been seated in a row, one pupil would have been without a seat. How many rows were there, and how many pupils?

Age Problems

8. Paul is one third as old as his father. Five years ago his father was four times as old as Paul. How old is each now?

9. Three years ago John was half as old as he will be 6 years from now. What is his present age?

10. Mary's mother said, "If you were 2 years older, you would be one third as old as Uncle Henry, who is now 36 years old." What is Mary's age?

11. William is three fourths as old as his brother Fred, but 3 years from now he will be five sixths as old as Fred. How old is each?

12. A father's age is 36, and that of his son is 3 years. In how many years will the father be four times as old as his son?

Coin Problems

13. James had \$1.65 in dimes and nickels. If there were 18 coins in all, how many of each kind did he have?

Suggestion. $10n + 5(18 - n) = 165$. (Explain.)

14. Henry changed a one-dollar bill in the grocery store, receiving in return 14 coins made up of dimes and nickels. How many of each kind did he receive?

15. I changed a five-dollar bill and received in return half dollars, quarters, and dimes. I received as many half dollars as quarters. If there were 17 coins in all, how many of each kind did I receive?

16. During a certain month my expense for postage was \$1.40. I sent only letters at 2 cents each and postcards at 1 cent each. How many of each kind did I send if I mailed in all 90 letters and postcards?

Geometric Problems

17. A line which is 12 inches long is to be divided into three parts such that each is 1 inch longer than the preceding part. Find the length of each part.

18. A line 70 feet long is to be divided into five parts such that each part is 2 feet longer than the preceding part. Find the length of each part.

19. The length of a rectangle is twice the width. If each dimension were increased by 5 feet, the perimeter would be 80 feet. Find the dimensions.

20. The side of a square is x. If the side were $x + 1$, the area would be 17 square units larger. Find x.

21. The altitude of a triangle exceeds the base by 4 inches. If the altitude is increased by 2 inches, the area is increased by 10 square inches. Find the base and the altitude.

22. Two angles of a triangle are equal. The third angle is twice as large as either of the other two. How many degrees are there in each of the three angles?

EXPLANATION. It can be proved that the sum of the angles of a triangle is 180 degrees.

23. One angle of a triangle is 87 degrees. The second angle is half as large as the third. How many degrees are there in each angle of the triangle?

24. If the sizes of the three angles of a triangle, measured in degrees, are denoted by consecutive integers, how many degrees are there in each angle of the triangle?

126. Summary. In this chapter you have learned

(1) How to solve an equation by interchanging members.
(2) How to transform an equation by addition or subtraction.
(3) How to solve equations with negative roots.
(4) How to transform an equation by changing signs.
(5) How to solve an equation involving parentheses.
(6) How to solve equations with fractional coefficients.
(7) How to solve literal equations.
(8) How to solve percentage problems, motion problems, mixture problems, and other applied problems.

Test on Chapter VII

The equations appearing in the second column below were obtained by changing the corresponding equations in the first column by one of these three methods: (I) *multiplying by* -1; (II) *interchanging members;* (III) *transforming by addition or subtraction. Indicate the method used in each case by writing I, II, or III.*

1. $7n-4=8+5n.$	$8+5n=7n-4.$
2. $7-2x=x+8.$	$-2x-x=8-7.$
3. $6a-10=2+3a.$	$-6a+10=-2-3a.$

Transform the equation $8y - 3 = 12y + 21 - 2y$, *as follows:*

4. Using Method I. 5. Using Method II.

6. Using Method III.

Solve each of the following equations:

7. $4x - (x + 12) = 2(21 - 3x)$. 10. $-7a + 4 = -3a - 8$.

8. $\frac{n}{3} + \frac{5n}{6} = \frac{7}{2}$. 11. $\frac{w+1}{2} = w - 12$.

9. $-4x - 15 = 9$. 12. $5b - y = 8b + 3 - b$.

13. In the formula $P = 2(l + w)$, solve for w.

Below each of the following problems you will find three equations. Select the one which correctly represents the conditions of the problem.

14. The length of a picture frame is 8 inches less than three times its width. If the perimeter is 32 inches, what are its dimensions?

(1) $2(4w - 8) = 32$. (2) $4w - 8 = 32$. (3) $2(3w - 8) = 32$.

15. A man sold his car for \$576, thereby losing 40 per cent of the original cost. How much did he pay for the car originally?

(1) $x - 40 = 576$. (2) $x - \frac{2x}{5} = 576$. (3) $x + .40 = 576$.

Exercises. *Review*

1. Given that $A = 6n^2 + 7n - 5$, $B = n^2 - 5n - 3$, and $C = 2n - 1$, find

(1) $A + B + C$; (3) $B \times C$;
(2) $C - B$; (4) $A \div C$.

Check the result obtained in each case.

2. Remove the parentheses and combine the similar terms in each of the following:

(1) $3\,a(a^2 - 5\,a - 8)$. (3) $10 - 7(1 + 4\,n - s) + 3\,n$.

(2) $7\,x - 2(x - 3)$. (4) $5\,n(2\,n - 3) - (12 + 7\,n^2)$.

3. Multiply $-6\,xy^3$ in succession by 5, $3\,x^3$, $-2\,xy$.

4. Divide $36\,a^3b^5c$ in succession by -9, $+3\,c$, $-4\,ab^2$.

5. In a certain city the monthly cost of electricity may be found by the formula $c = 1.25 + .04\,e$, in which e represents the amount of electricity used and c the cost in dollars. Find the value of c when $e = 60$.

6. The formula for the tension in a cable is $T = 3 + \frac{wa}{g}$.

(1) Find T when $w = 200$, $a = 12$, and $g = 32$.

(2) Find a when $T = 1875$, $w = 1200$, and $g = 32$.

7. The average amount of life insurance sold in Canada per day, in hundreds of thousands of dollars, during eight successive years, beginning in 1927, was as follows: 23, 25, 27, 24, 21, 18, 16, 16.

(1) Draw a graph representing these statistical facts.

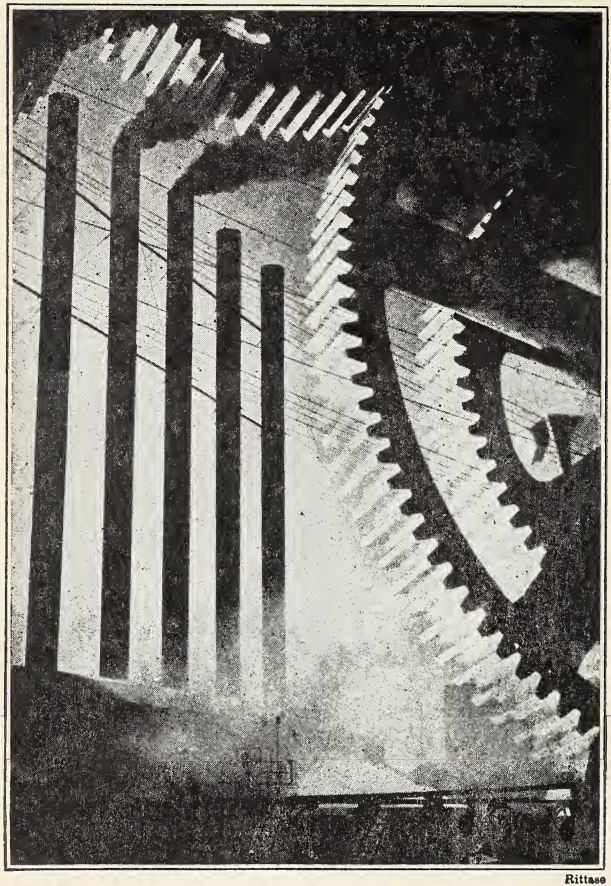
(2) Indicate by a line the average daily sales for the six-year period.

(3) Indicate by means of signed numbers how the figure given for each year varies from the average.

8. The cost of sending a telegram to a certain city, for messages of various lengths, is shown in the following table:

Number of words	10	14	20	24	30	40
Cost in cents	42	52	67	77	92	117

Find a formula for the cost of a telegram of ten or more words. (Use c and w.)



Rittase

This photograph symbolizes the important part that geometry plays in industry. Geometry is used in building roads and bridges, in surveying land, in constructing railways, and in countless other ways. Directly or indirectly geometry is vital to us all

Part Two · *Geometry for Today*

CHAPTER VIII

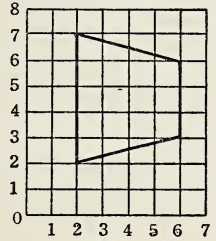
FIRST STEPS IN GEOMETRY

127. First Steps in Geometry. In this chapter you may review briefly the more important facts you learned in earlier grades about lines, angles, and plane figures. It is not intended that the chapter be studied in its entirety; your teacher will tell you which parts you require.

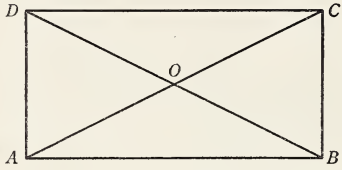
128. Measurement of Lines and Angles. You will need a ruler and protractor in the following exercises.

Exercises. *Measuring Lines and Angles*

1. On squared paper draw the figure shown below. Measure each angle of this figure. Find the sum of these angles.



2. Measure the lines BC, BO, OC, and BA. Give each length to the nearest one-sixteenth inch or one-tenth inch.



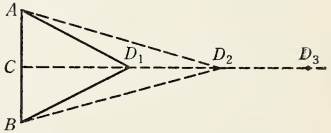
3. Name by letter, from the above figure, an example of each of the following:

(1) A right angle (an angle of 90°).
(2) An acute angle (an angle less than 90°).
(3) An obtuse angle (an angle greater than 90°).
(4) A straight angle (an angle of 180°).

4. Classify each of the following angles as acute, right, obtuse, or straight:

120° 40° 180° 150° 90° 85° 105° 75° 135°

5. Stick two pins into a board as at A and B. Stretch an elastic band into position ABD_1. Let D move back and forth along the line CD so that DA is always equal to DB.

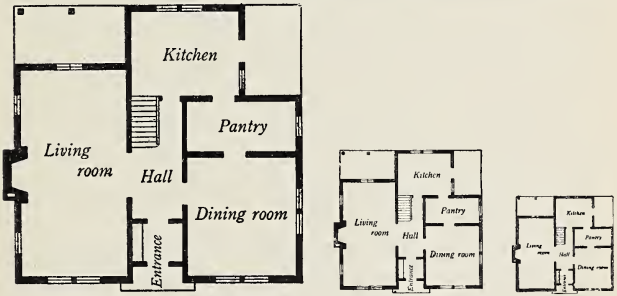


(1) What change takes place in the angles A and B as D_1 moves out to D_2? to D_3? beyond D_3?

(2) What change takes place in the angle D for each different position of D?

(3) Imagine that D could be stretched out 1 mile from C, then 100 miles from C, and then farther. Show how angles A, B, and D change as D gets farther from C.

129. Scales and Scale Drawing. When an architect designs a house, he cannot draw the plans "full size." Instead he draws a *reduced* picture of the various parts of the house, very much as a photograph made by an ordinary camera is smaller than the original object which it represents. To make such drawings the architect must, first of all, choose a convenient *scale*. For example, he may let



$\frac{1}{4}$ of an inch represent 1 foot. This is written in various forms, such as "Scale $\frac{1}{4}$ in. to 1 ft.," or "Scale $\frac{1}{4}''$ to $1'$," or "Scale $\frac{1}{4}$ in. $=$ 1 ft."

Having selected a suitable scale, the architect must *change the actual dimensions of the house to the corresponding scale lengths.*

Thus, suppose that a house is to be 32 feet long and 24 feet wide. If the scale is $\frac{1}{4}$ inch to 1 foot, the plan of the house must be 8 inches long and 6 inches wide.

When the entire plan of a house is thus drawn correctly to scale, a contractor or workman using a blue print of such a plan can readily *change the dimensions of the blue print back to the true dimensions* and so do the actual work suggested by the plan.

Exercises. *Drawing to Scale*

1. A house is 24 feet wide and 32 feet long. Draw the ground plan of the house, letting $\frac{1}{8}$ inch represent 1 foot.

SUGGESTION. If $\frac{1}{8}$ inch represents 1 foot, the scale lengths of the plan are readily seen to be 3 inches and 4 inches. Hence, on squared paper draw a rectangle 3 inches wide and 4 inches long.

2. The width of a house is 32 feet and its length is 40 feet. Draw a plan of the house, letting $\frac{1}{8}$ inch represent 1 foot.

3. A schoolroom is 20 feet wide and 24 feet long. Draw a plan of the room, letting $\frac{1}{8}$ inch represent 1 foot.

4. A window is 3 feet wide and 9 feet high. Draw a plan of the window, letting $\frac{1}{4}$ inch represent 1 foot.

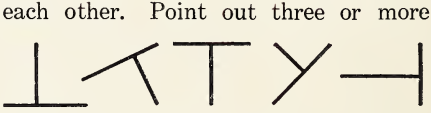
5. A building lot is 50 feet wide and 150 feet long. Draw a plan of the lot, letting 1 inch stand for 25 feet.

6. If possible, find a magazine which contains plans of houses. Select one of these plans and copy it to a different scale.

130. Positions of Lines. The following exercises will help you to recall certain facts about the positions of lines.

Exercises. *Positions of Lines*

1. If two lines form a right angle, they are said to be *perpendicular* to each other. Point out three or more sets of perpendicular lines in the schoolroom.



2. Look at the rectangle suggested by the wall in front of you. The upper and lower edges of the wall are said to be *opposite* sides of this rectangle; as are the left and right

edges. Notice that the upper edge is everywhere the same distance from the lower edge. They are said to be *parallel.* This means that they run "alongside each other" without ever coming nearer together. Evidently this is true of the longer sides of *any* rectangle, and also of the shorter sides. That is, *every rectangle has two sets of parallel sides.* Again, *the opposite sides of every rectangle are equal.* Point out some sets of parallel lines in the room.

3. In Fig. 1 the two straight lines AB and CD intersect at an acute angle n.

Note the change in the angle n as CD swings about C as centre. In Fig. 2 the point D (end of line CD) is shown in various positions D_1, D_2, D_3, D_4, D_5, D_6, and D_7.

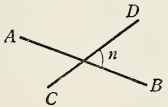


FIG. 1

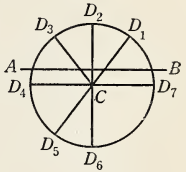


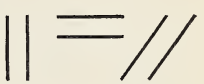
FIG. 2

(1) In which positions of CD (Fig. 2) do AB and CD appear to be perpendicular? How can you test their perpendicularity with the protractor? with the set square?

(2) In which positions of CD (Fig. 2) do AB and CD appear to be parallel? How would you test this?

(3) Name two acute angles and two obtuse angles.

4. Notice that parallel lines may be vertical, horizontal, or slanting. Illustrate these various positions with two sticks or pencils.



5. The hands of a clock, as they revolve, illustrate every possible kind of angle. At 3 o'clock the hands form a right angle and are therefore perpendicular to each other. Are they perpendicular at any other time?

131. Areas of Plane Figures. A plane figure is a figure which lies all in one flat surface; and a plane figure bounded by straight lines is known as a rectilinear figure, or polygon.

Area of a Rectangle. The area of a rectangle can be found by means of the formula $A = lw$, as on page 25.

Area of a Parallelogram. You know that a parallelogram is a quadrilateral (a four-sided polygon) whose opposite sides are parallel. From the figures below you see that the right triangle BCE, shown in Fig. 1, could be cut from the parallelogram $ABCD$ and placed as shown in Fig. 2, thus forming the rectangle $ABEF$. It is

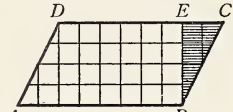


FIG. 1

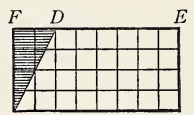


FIG. 2

therefore obvious that the area of the parallelogram $ABCD$ is equal to the area of the rectangle $ABEF$.

In either figure, AB is the base and BE is the height. Hence we have the following rule:

The area of a parallelogram is equal to the product of its base and its height.

NOTE. This rule may be written in the form $A = bh$. (Explain.)

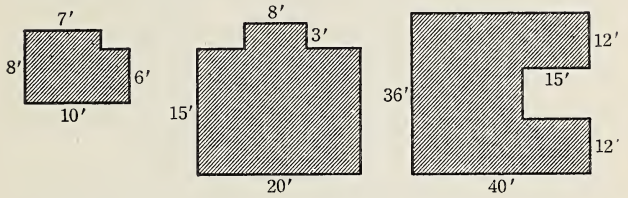
Area of a Triangle. The formula for the area of a triangle is $A = \frac{1}{2}\, bh$ (see page 28).

Area of a Circle. The area of a circle, as you learned on page 33, may be found by using the formula $A = \pi r^2$ ($\pi = 3\frac{1}{7}$ or 3.1416).

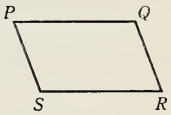
The exercises which follow will give you an opportunity to see how well you understand these formulas.

Exercises. *Finding Areas*

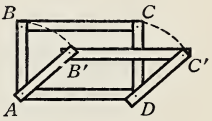
1. Find the area of each of the following figures:



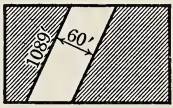
2. Use the rule $A = bh$ to find the area of the parallelogram $PQRS$, first using SR as the base and second using QR as the base.



3. Suppose that four sticks are hinged together so as to form a rectangular frame such as $ABCD$ in the figure. If you press down on the frame, you will cause it to take the form of a parallelogram such as $AB'C'D$ in the figure. In which position does the frame enclose the greatest possible area? Why?



4. A new automobile road was built across Mr. Kelly's farm, cutting out a parallelogram having the dimensions shown in the figure at the right.



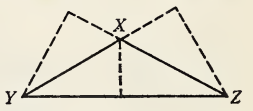
(1) How many square feet of his land did Mr. Kelly have to give up?

(2) How many acres of his farm did he give up?

(3) If he received $500 for each acre, how much money did he receive?

5. (1) Measure the sides of triangle XYZ.

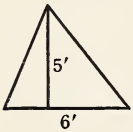
(2) The area A of a triangle is found by the formula $A = \frac{1}{2}\,bh$, where b is the length of the base and h is the perpendicular height. Find the area, first with YZ as base, second with XY as base, third with XZ as base. (Note that you use a different perpendicular each time. Why? Do you get the same result each time? Why?)



6. Repeat Exercise 5, drawing, in your notebook, a larger and differently shaped triangle.

Find the area of each of the following triangles, using the given dimensions:

7.



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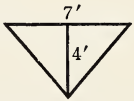
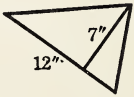


9.

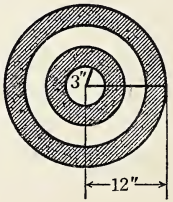


10. On squared paper draw a circle with radius 1 inch in length. Find the approximate area of the circle

(1) By counting the small squares included within the circle.

(2) By using the formula $A = \pi r^2$ (take $\pi = 3\frac{1}{7}$).

11. The target shown at the right has four concentric circles drawn on it. The smallest of the circles has a radius of 3 inches, while the largest circle has a radius of 12 inches. The area of the largest circle is _(?)_ times the area of the smallest.



132. How to Bisect a Given Line.* **(Construction 1.)** Suppose that AB is the line-segment which is to be bisected. We proceed as follows:

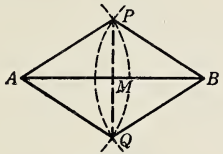


FIG. 1

FIG. 2

1. Using A as a centre and any convenient radius (see page 32) greater than $\frac{1}{2}\,AB$, we draw an arc† of a circle.

2. Then, using B as a centre and the *same* radius, we draw an arc of a circle cutting the first arc at the points P and Q.

3. We join P and Q. Then the line PQ bisects AB at M.

The accuracy of the construction may readily be tested with the compasses or with a ruler.

In practice, it is sufficient simply to locate the two points P and Q by drawing only portions of the two arcs, as shown in Fig. 2 above. Observe that *three steps* are sufficient to bisect AB. You should be able to bisect a given line-segment in 15 seconds.

QUIZ: 1. What is meant by "any convenient radius"?

2. Why is the radius made greater than $\frac{1}{2}\,AB$?

3. What is the longest radius that might be used?

4. How could you divide a line into four equal parts?

* Any drawing of a straight line actually represents but a small portion of that line. That is, we may *imagine* the line extended indefinitely in either direction. Any part of a straight line lying between two points on that line is called a *segment* of that line. The word "line" is often used in the sense of "line-segment," and it will be so used hereafter unless there is danger of confusion.

† An *arc* of a circle is any part of the circumference.

133. How to Bisect a Given Angle. (Construction 2.) Suppose that $\angle P$ (angle P) in Fig. 1 is the angle to be bisected.

FIG. 1

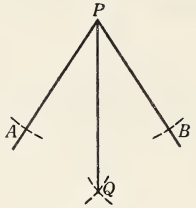


FIG. 2

1. With P as centre and with any convenient radius, draw an arc of a circle cutting one side of the angle at A and the other at B.

2. With A as centre and radius greater than $\frac{1}{2}\ AB$, draw an arc.

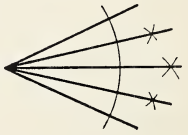
3. With B as centre, and with the *same* radius, draw a second arc cutting the first arc at Q.

4. Join P and Q. Then the line PQ bisects the given angle.

Exercises. *Bisecting an Angle*

1. Draw an acute angle. Bisect the angle and also each of the resulting halves, as suggested by the figure below.

2. Explain how to divide an angle into 4, 8, 16, · · · parts.



3. If the angle is 50°, what is the size of the angles formed by repeated bisections?

4. Draw a straight angle, naming it $\angle AOB$.

(1) Construct the bisector of $\angle AOB$, naming it OC.

(2) What kind of angle is $\angle AOC$? $\angle BOC$?

134. How to Construct a Perpendicular to a Given Line at a Point in the Line. (Construction 3.) Given a line RS and a point M on that line. We may construct a perpendicular to RS at M as follows:

1. With M as centre and any radius less than MR (or MS, whichever is the shorter), draw an arc of a circle cutting RS at A and B.

2. Locate the points P and Q as you did in section 132.

3. Join PM. PM is the required perpendicular.

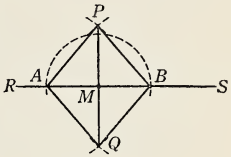


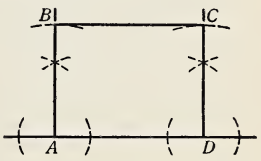
FIG. 1

NOTE. The symbol $\perp$ is often used in place of the word "perpendicular" or of the phrase "is perpendicular to." Thus, the statement "$PQ \perp RS$" means that PQ is perpendicular to RS. The plural form, "perpendiculars," is indicated by the symbol $\perp$s.

Exercises. *Using Perpendiculars*

1. Construct a rectangle which is 4 inches × 3 inches.

Solution. On a straight line lay off $AD = 4$ inches. At A and at D construct perpendiculars to AD. Make $AB = DC = 3$ inches. Draw BC. Then $ABCD$ is the required rectangle.

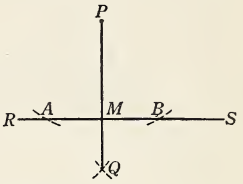


2. Construct a rectangle which is 3 inches × 2 inches.

3. Draw a triangle. Then construct the perpendicular bisector of each of its sides. What do you observe? Test the truth of your observation by drawing several triangles.

4. Construct a square on a side of $3\frac{1}{2}$ inches.

135. Constructing a Perpendicular to a Given Line from a Given Point outside of the Line. (Construction 4.) Given a line RS and a point P outside of the line. To construct a perpendicular to RS from P, we proceed as follows:



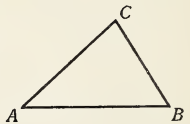
1. Using P as a centre and a radius of sufficient length, we draw an arc cutting RS at A and B.
2. Locate the point Q as in section 134.
3. Then PM is the required perpendicular.

Exercises. *Constructing Perpendiculars*

1. If different radii had been used in the above construction, would you have obtained a different perpendicular?

2. On a straight line lay off the successive equal segments AB, BC, CD, DE, and EF. Construct perpendiculars to the given line at the points B, C, D, and E.

3. Draw a triangle ABC as shown, but larger. Construct a perpendicular from each vertex to the side opposite.



4. Draw a triangle DEF as shown and construct its altitudes.

NOTE. An *altitude* of a triangle is a perpendicular drawn from any vertex to the opposite side.

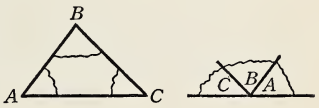
136. The Angle-Sum of a Triangle. Whenever you change the size or the shape of a triangle, in general the *perimeter* will also change. It is an interesting question whether this is also true of the sum of the *angles*. The exercises which follow will enable you to make a surprising discovery.

Exercises. *The Angle-Sum of a Triangle*

1. Draw four large triangles of different shapes and sizes. With your protractor measure the angles of each triangle. Then find the sum of the angles of each triangle. Arrange your work in a table, like the one at the right.

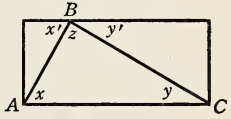
Angle	△ 1	△ 2	△ 3	△ 4
A . . .				
B . . .				
C . . .				
Sum				

2. If the work suggested in Exercise 1 is done carefully, you will find that *the sum of the angles*, in each triangle, *appears to be 180°* (very nearly). To test the accuracy of this conclusion, cut a triangle out of paper, tear off the corners, and place them as shown in the adjoining figure. Again it seems to be true that



$$\angle A + \angle B + \angle C = 180°.$$

3. Another simple way of verifying the conclusion suggested by Exercise 1 is indicated in the figure shown below. Take a rectangular sheet of paper, and draw the triangle *ABC* as shown in the figure. Cut out the triangle. You will find that $\angle x = \angle x'$, and that $\angle y = \angle y'$.



But $\angle x' + \angle y' + \angle z = 180°$. (Why?)

Hence $\angle x + \angle y + \angle z = 180°$.

4. If you know the number of degrees in *two* angles of a triangle, how can you find the number of degrees in the *third* angle without measuring it?

5. A surveyor measured two angles of a triangular corner lot. He found them to be 62° and 63°. Did the surveyor have to measure the third angle, to be sure of its size? How large was the third angle?

In each of the following exercises two angles of a triangle are given. Find the third angle.

6. 30°, 70°.	8. 40°, 50°.	10. 100°, 70°.	12. 120°, 30°.
7. 80°, 50°.	9. 60°, 60°.	11. 90°, 60°.	13. 45°, 45°.

137. Special Names for Triangles with Angles of Different Sizes. Triangles are frequently referred to by special names which depend on the sizes of the angles of the triangle. It is important that you learn these names.

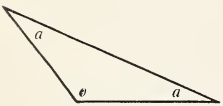


FIG. 1

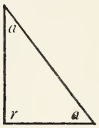


FIG. 2

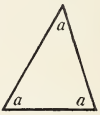


FIG. 3

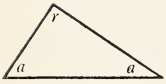


FIG. 4

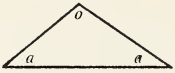


FIG. 5

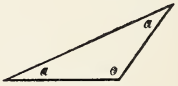


FIG. 6

In these six triangles all *obtuse angles* are marked *o*, *acute angles* are marked *a*, and *right angles* are marked *r*.

Figs. 1, 5, and 6 are called *obtuse-angled* triangles because they each contain an obtuse angle.

Figs. 2 and 4 are called *right-angled* triangles because they each contain a right angle.

Fig. 3 is called an *acute-angled* triangle because *all* the angles are acute.

Exercises. *Naming Triangles by their Angles*

1. Copy the following table and fill in the blanks with the correct number of angles in each instance:

	Type of Triangle	Number of Acute Angles	Number of Right Angles	Number of Obtuse Angles
(1)	Acute-angled .			
(2)	Right-angled .			
(3)	Obtuse-angled .			

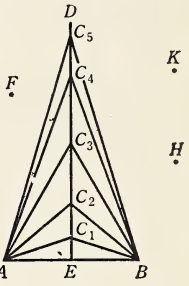
2. Draw each of the following triangles, using the given measurements. Tell whether each is acute-angled, right-angled, or obtuse-angled.

	AB	BC	CA	$\angle A$	$\angle B$	$\angle C$
(1)	2 in.			40°	50°	
(2)		3 in.			70°	60°
(3)			$1\frac{3}{4}$ in.	100°		20°
(4)	3 in.	4 in.			60°	
(5)	$2\frac{1}{2}$ in.		$2\frac{1}{2}$ in.	90°		
(6)		5 cm.	7 cm.			135°

3. If an obtuse-angled triangle contains an angle of 120°, what is the sum of the other two angles?

4. If one acute angle of a right-angled triangle contains 16°, how many degrees are there in the other acute angle?

138. Special Names for Triangles Determined by the Lengths of the Sides. In the figure, E is the mid-point of AB. ED is perpendicular to AB. Each point on ED is equally distant from A and B. Triangles ABC_1, ABC_2, ABC_3, etc. are called *isosceles* triangles (triangles in which two sides are equal).



Going along ED from E to C_1, C_2, etc., we see that the sides AC_2 and BC_2 are longer than AC_1 and BC_1. Notice also that AC_1, BC_1, AC_2, and BC_2 are all shorter than AB, the base, but that AC_5 and BC_5 are each longer than AB. Somewhere on ED there is one point, C_3 in this case, that is the same distance from A and B that A is from B. The triangle ABC_3 is *equilateral* (it has all its sides equal).

If AB were the base while the vertex was at F, or H, or K, or at any point not on ED, the triangle would be *scalene* in most instances (no two sides equal). An exception occurs when an isosceles triangle with AB as one of the equal sides is formed.

Exercises. *Kinds of Triangles*

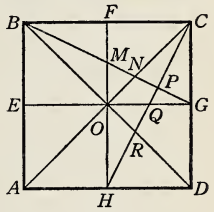
1. If the vertex angle (angle between the equal sides) of an isosceles triangle has the value given in the exercises below, find in each instance the number of degrees in one of the base angles:

(1) 80°	(4) 60°	(7) 30°
(2) 100°	(5) 90°	(8) 150°
(3) 120°	(6) 72°	(9) 36°

2. If one of the base angles of an isosceles triangle has the value given below, how large is the vertex angle?

(1) 40° (3) 60° (5) 72° (7) 54°
(2) 50° (4) 45° (6) 30° (8) $67\frac{1}{2}°$

3. In the square $ABCD$, the diagonals and mid-lines were drawn, also the lines BG and CH. In this figure name



(1) Two right-angled triangles.
(2) Two isoseeles triangles.
(3) Two obtuse-angled triangles.
(4) Two scalene triangles.
(5) Two acute-angled triangles.
(6) Two straight angles.

139. Meaning of Congruence. Place several sheets of paper one on top of the other. On the outer sheet draw any simple pattern. Then cut out this pattern from the entire set of papers.

Place all the resulting figures side by side, and it will then be seen that these figures *agree in all respects.* Such figures are said to be *congruent* (from the Latin *congruere,* "to agree").

Two figures are said to be congruent if one is an exact copy of the other, or if both are exact copies of the same figure. Congruent figures *can be made to fit on each other,* or to *coincide.* That is, *they have the same shape and the same size.*

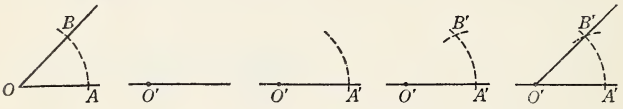
Note the following everyday examples of congruence:

1. Copies of the daily newspaper.
2. All photographs printed from one negative.
3. Carbon copies of typewritten letters.
4. Two shoes, of the same style, made on the same last.
5. Dress parts cut from the same pattern.

140. Corresponding Parts of Congruent Figures. In general, we may say that *the corresponding parts of two congruent figures are those parts which exactly coincide when one figure is placed on top of the other figure.*

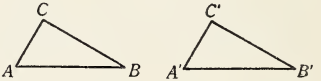
The corresponding parts of two congruent figures are often indicated by using corresponding *check marks*, such as those shown in the diagram above, or by checking the corresponding sides with crayons of different colors.

141. Constructing an Angle Equal to a Given Angle. (Construction 5.) The figures below suggest the *five* steps which are necessary in constructing an angle equal to a given angle. Explain each of these steps used in making $\angle A'O'B' = \angle AOB$.



142. Constructing Congruent Triangles. If two triangles, such as *$\triangle ABC$ and $\triangle A'B'C'$, are to be congruent, their corresponding sides must be equal and their corresponding angles must also be equal. That is, the following *six* equalities must be true:

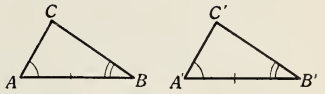
$AB = A'B' \quad \angle A = \angle A'$
$BC = B'C' \quad \angle B = \angle B'$
$AC = A'C' \quad \angle C = \angle C'$



But it is possible to copy $\triangle ABC$ by using only three of these equalities. This may be done in several ways.

* The sign $\triangle$ is often used in place of the word *triangle*.

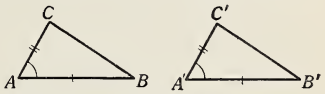
First Method (*a.s.a.*). Construct $A'B'$ equal to AB, $\angle A'$ equal to $\angle A$, and $\angle B'$ equal to $\angle B$. Then the sides of $\angle A'$ and $\angle B'$, when extended, will intersect at a point C', which corresponds to C.



It is easy to see that $\triangle A'B'C'$ will fit on $\triangle ABC$. This fact could be verified by drawing $\triangle A'B'C'$ on tracing paper and placing it on $\triangle ABC$. It will then be seen *that the other three parts of* $\triangle A'B'C'$ *are also equal to the corresponding parts of* $\triangle ABC$, *namely, that* $A'C' = AC$, $B'C' = BC$, *and* $\angle C' = \angle C$.

Since we copied *by construction* only *angle A*, *side AB*, and *angle B*, this method is known as the method of "angle, side, angle," or, more briefly, as the method of *a.s.a.*

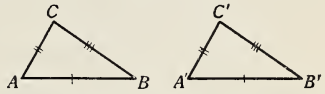
Second Method (*s.a.s.*). Construct $A'B'$ equal to AB, $\angle A'$ equal to $\angle A$, and make $A'C'$ equal to AC. Then the side $B'C'$ can be drawn at once, thus completing $\triangle A'B'C'$.



The congruence of the two triangles can be tested exactly as in the case of the preceding method. Hence it follows *that the other three parts of* $\triangle A'B'C'$ *are also equal to the corresponding parts of* $\triangle ABC$, *namely, that* $B'C' = BC$, $\angle B' = \angle B$, *and* $\angle C' = \angle C$.

Since we copied *by construction* only *side AB*, *angle A*, and *side AC*, this method is known as the method of "side, angle, side," or, more briefly, as the method of *s.a.s.*

Third Method (*s.s.s.*). Make $A'B'$ equal to AB. Using A' as a centre and a radius equal to AC, draw an arc. Then, with B' as a centre and with a radius equal to BC, draw an arc. These two arcs will intersect at a point C'. It is then possible to complete $\triangle A'B'C'$ by drawing the sides $A'C'$ and $B'C'$.



Exercises. *Triangles*

1. Draw $\triangle ABC$, if $AB = 3$ inches, $\angle A = 40°$, $\angle B = 60°$. Make $\triangle A'B'C'$ congruent to it.

2. Draw $\triangle DEF$, if $EF = 2\frac{1}{2}$ inches, $\angle E = 55°$, and $\angle F = 70°$. Copy the triangle.

3. Draw $\triangle MNP$, in which $MN = 2$ inches, $MP = 3$ inches, and $\angle M = 45°$. Copy the triangle.

4. Draw $\triangle BCD$, in which $CD = 1\frac{7}{8}$ inches, $BC = 2\frac{1}{4}$ inches, and $\angle C = 75°$. Copy the triangle.

5. Construct $\triangle ABC$, in which $AB = 2$ inches, $AC = 3\frac{1}{2}$ inches, and $BC = 3$ inches. Copy the triangle.

6. Construct $\triangle RST$, in which $RS = 5$ centimetres, $ST = 6$ centimetres, and $RT = 7$ centimetres.

7. Find all the angles of an isosceles triangle if the vertex angle is 25°.

8. The angles of a triangle are $x°$, $3x°$, and 40°. Find the unknown angles.

9. One of the angles of a right-angled triangle is 52°. Find the other acute angle.

10. The perimeter of an equilateral triangle is 45 inches. Find the length of each side.

11. The length of each of the equal sides of an isosceles triangle is twice that of the base. If the perimeter is 40 inches, find the length of each side.

12. Write a sentence using correctly each of the following:

perimeter	isosceles	equilateral
polygon	segment	scalene
obtuse	perpendicular	parallelogram
formula	congruent	vertex

143. Summary. In this chapter you have learned or reviewed

(1) How to measure lines and angles.
(2) How to draw to scale.
(3) Important formulas for the areas of plane figures.
(4) How to bisect a given line.
(5) How to bisect a given angle.
(6) How to construct perpendiculars.
(7) Important facts about the angles and sides of a triangle.
(8) The meaning of congruence.

Test on Chapter VIII

1. Classify the following angles as acute, right, obtuse, or straight: 130°, 55°, 100°, 90°, 45°, 180°.

2. Write the formulas for the area of a triangle, the area of a parallelogram, and the area of a circle.

3. Draw a line of any convenient length. Without measuring it, divide it into four equal parts.

4. Draw an obtuse angle. Without measuring it, divide it into four equal parts.

5. Construct a rectangle which is $3\frac{1}{2}$ inches $\times$ $2\frac{1}{2}$ inches.

6. Explain what you mean by an obtuse-angled triangle; a scalene triangle; a right-angled triangle; an isosceles triangle; an equilateral triangle.

7. Draw $\triangle PQR$, given $QR = 3\frac{1}{2}$ inches, $\angle Q = 45°$, $\angle R = 35°$. Construct $\triangle P'Q'R'$ congruent to it.

8. Construct $\triangle XYZ$, given $XY = 1\frac{1}{2}$ inches, $YZ = 2$ inches, $ZX = 2\frac{1}{2}$ inches. Measure $\angle Y$.

9. Draw $\triangle MNP$, given $MN = 3\frac{1}{4}$ inches, $NP = 2\frac{1}{2}$ inches, and $\angle N = 45°$. Copy the triangle.

CHAPTER IX

ANGLES

1. Necessary Words Explained

144. The Content of Geometry. Your first work in geometry consisted largely of drawing and measuring lines, angles, and plane figures. You learned later how to construct certain figures with straightedge and compasses rather than draw them with ruler and protractor. However, when you were bisecting lines and angles, constructing perpendiculars, finding the angle-sum of a triangle, or constructing congruent triangles, you were not asked to consider whether your reasoning for the figure before you would be suitable for *all* figures of the same type or whether preceding propositions would give the necessary basis for this proposition. You were not asked to give what we call "proofs" nor asked to show in detail the logical relations. In the following chapters you will be introduced gradually to these proofs.

A statement submitted for consideration is often called a *proposition.* For example, *The sum of the three angles of a triangle is equal to two right angles* is a proposition. A proposition such as this is called a *theorem* for the reasons given in sections 160 and 174. *To bisect a given angle* is another kind of proposition. Here there is a construction to be completed. One must prove that the construction suggested actually gives the desired results.

145. Meaning of *Theorem*. Theorems often begin with the word *If*. When otherwise stated they may be thought of in the form *If it is true that . . . , then it is true that . . . .* The *If* part is assumed or *given*; the part to be proved is called the *required to prove* part. The *If* part is sometimes called the *hypothesis*; the part to be proved, the *conclusion*.

In the theorem *An equiangular triangle is also equilateral,* the given part is *If the three angles of a triangle are equal,* and the required part is *the three sides are then equal.*

Exercises. *Meaning of Theorems*

State what is given and what is required in each of the following theorems:

1. If the opposite sides of a quadrilateral are equal, the opposite angles are equal.

2. The line bisecting the vertex angle of an isosceles triangle bisects the base.

3. If the four sides of a quadrilateral are equal, the diagonals bisect each other.

4. The opposite sides of a parallelogram are equal.

Many theorems are such that we are willing, at this stage, to accept without question the truths which they express. From past experience we believe them to be true. For example, *When one straight line meets another straight line the sum of the adjacent angles is equal to two right angles* is accepted without proof in section 148. From our experience with lines and angles we believe that this statement is always true. We do not ask for a formal proof. Further on in this book you will meet theorems for which the *then it is true that . . .* part is not so evident. You will be asked to prove these theorems.

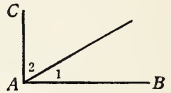
146. Meaning of *Problem*. In a problem a definite construction is to be made. Sometimes there is no figure and one meeting certain conditions must be constructed. In the problem *Construct an equilateral triangle* no figure is given. The figure to be drawn must have three equal sides.

In some problems there is a given figure and additional constructions are to be made. For example, *Construct a triangle equal in area to a given parallelogram.* Here there is a given parallelogram. The additional figure required is a triangle equal in area to the parallelogram.

In Chapter VIII five fundamental problems were completed without proofs.

2. Angles at a Point

147. Complementary Angles. If the sum of two angles is a right angle, the angles are called *complementary* angles. $AC \perp AB$ and therefore $\angle 1 + \angle 2 = 90°$. $\angle 1$ and $\angle 2$ are the complements of each other.



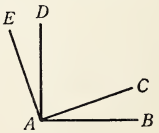
Exercises. *Complementary Angles*

1. Name three pairs of complementary angles, giving their values in degrees.

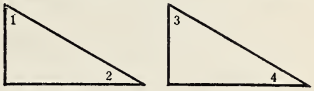
Note. Two angles having the same vertex and a common side *between* them are called *adjacent angles*, as $\measuredangle BAC$ and CAD in the figure below.

2. Draw an acute angle and construct its adjacent complement. Show how to do this in two ways.

3. In the figure at the right, if it is true that $DA \perp AB$ and $EA \perp AC$, find two pairs of angles that are complementary.



4. If it is true that $\angle 1 + \angle 2 = 90°$ and $\angle 3 + \angle 4 = 90°$ and $\angle 1 = \angle 3$, what conclusion may you draw? If $\angle 1 = 58°$, $\angle 4 =$ _(?)_.



If there are two equal angles A and B, it is true that $90° - \angle A = 90° - \angle B$. This equation does not depend upon the values that $\angle A$ and $\angle B$ may take so long as they remain equal.

THEOREM 1

Complements of the same or of equal angles are equal.

5. The angles X and 40° are complementary. What is the value of $\angle X$?

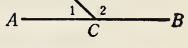
6. If one angle be D, the formula for the number of degrees in its complement, C, is $C = 90 - D$. Find $\angle C$ when $\angle D$ is $18\frac{1}{2}°$.

7. If $\angle X = \angle Y$, then the complement of $\angle X =$ _(?)_.

148. Supplementary Angles. If the sum of two angles is 180°, the angles are *supplementary* and each is the supplement of the other.

$$\angle ACB = 180°.$$

Hence $\angle 1$ and $\angle 2$ are supplementary.

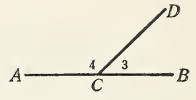


THEOREM 2

When one straight line meets another straight line, the sum of the adjacent angles is equal to two right angles.

Exercises. *Supplementary Angles*

1. If AC, DC, and BC meet at C and if it is true that $\angle 3 + \angle 4 = 180°$, then $\angle ACB$ is a straight angle and AC and CB lie in one straight line. Test the truth of this by assigning values to $\angle 3$ and $\angle 4$ such that $\angle 3 + \angle 4 = 180°$.



2. If one angle be $D°$, the formula for the number of degrees in its supplement is

$$S = 180 - D.$$

Find $\angle S$ if $\angle D$ is $90°$; $12°$; $175°$.

3. $\angle A$ and $\angle B$ are supplementary. $\angle A$ is $30°$ greater than $\angle B$. How many degrees are there in each angle?

4. The angles X and $42°$ are supplementary. Find X.

5. State by an equation that X and Y are supplementary angles.

If there are two equal angles A and B, it is true that

$$180° - \angle A = 180° - \angle B.$$

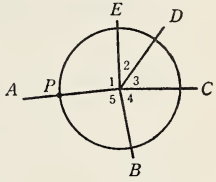
THEOREM 3

Supplements of the same angle or of equal angles are equal.

6. If $\angle A$ is the supplement of $\angle B$ and $\angle B$ is the supplement of $\angle C$, then $_(?)_ = _(?)_$.

7. If an acute angle contains $23°$, what is the difference, in degrees, between its supplement and its complement?

149. Sum of the Angles at a Point. Starting at any point P on the A line and going in a circle in a clockwise direction, that is, as the hands of a clock travel, one will arrive back at P. In completing the circle one will turn through the angles 1, 2, 3, 4, and 5 in succession. The sum of these angles must be 360°.



THEOREM 4

If any number of straight lines meet at a point, the sum of all the adjacent angles is equal to four right angles.

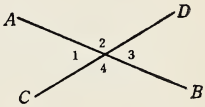
150. Vertically Opposite Angles. Two angles are *vertically opposite* when each may be formed by producing the arms of the other beyond the vertex.

AB and CD are any two intersecting straight lines. $\angle 1$ and $\angle 3$ are vertically opposite; so are $\angle 2$ and $\angle 4$.

$\angle 2$ is the supplement of $\angle 1$.

$\angle 4$ is the supplement of $\angle 1$.

Therefore $\angle 2 = \angle 4$, because supplements of the same angle are equal (Theorem 3).



In the same manner we arrive at these facts:

$\angle 3$ is the supplement of $\angle 2$.

$\angle 1$ is the supplement of $\angle 2$.

Therefore $\angle 3 = \angle 1$.

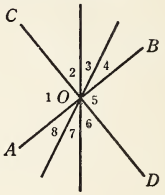
THEOREM 5

If two straight lines intersect, the vertically opposite angles are equal.

Exercises. *Angles*

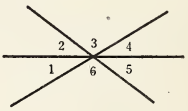
1. In the accompanying figure four straight lines intersect in O. Two of these lines, AB and CD, are perpendicular to each other.

Complete in your notebook the following statements:



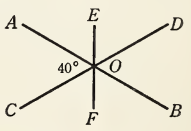
(1) $\angle 4 + \angle 5 =$ _(?)_.
(2) $\angle 7 +$ _(?)_ $+$ _(?)_ $= 90°$.
(3) The complement of $\angle 3$ is _(?)_.
(4) $\angle 1 = \angle$ _(?)_.
(5) $\angle 2 + \angle 3 = \angle 6 + \angle$ _(?)_.
(6) $\angle 2 + \angle 3 + \angle 4 + \angle 5 + \angle 6 + \angle 7 + \angle 8 =$ _(?)_.
(7) $\angle 5 =$ _(?)_.
(8) $\angle 1 + \angle 2 + \angle 3$ is the supplement of _(?)_.

2. Find the value of $\angle 2 + \angle 4 + \angle 6$ in the figure below at the right.



3. If three straight lines intersect in a point, how many pairs of vertical angles are formed? How many of these angles must be measured in order that you may know the sizes of all?

4. If 30 straight lines intersect in a point, how many pairs of vertical angles are formed?

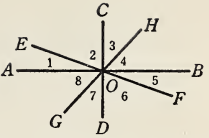


5. The angle AOD is bisected by EF. Find the value of each angle shown in the figure.

6. Find X if $X° + 10°$ is the supplement of $X° + 30°$.

7. Two straight lines intersect in the point O, making four angles, one being $a°$ and another $5\ a°$. Find the value of a.

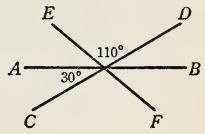
8. In the figure at the right, $AB \perp CD$ and EF and GH are straight lines. Which of the angles 2, 3, 4, 5, 6, 7, and 8 are of known value if $\angle EOA$ is 20°?



9. An angle of $X°$ is complementary to an angle of $4\ X°$. How many degrees are there in each angle?

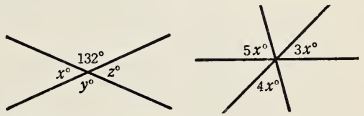
10. The supplement of an angle of $Y°$ is $\frac{1}{4}\ Y°$. Find the value of each angle.

11. In a $\triangle ABC$, $\angle B = \angle C$. BC is produced to D. CB is produced to E. Draw the figure and show that $\angle ABE = \angle ACD$.



12. Find the value of each angle in the accompanying figure if AB, CD, and EF are straight lines.

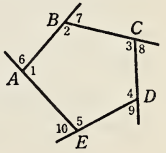
13. Find the value of each angle in the two figures at the right.



14. When any side of a triangle is produced in either direction it forms an exterior angle of the triangle. Draw any triangle. Write I on each interior angle. Write E on each exterior angle. How many interior angles are there? How many exterior angles are there?

15. In the polygon $ABCDE$ the angles 1, 2, 3, 4, and 5 are *interior* angles. The *exterior* angles 6, 7, 8, 9, and 10 are formed by producing EA, AB, BC, CD, and DE.

If the sum of angles 1, 2, 3, 4, and 5 is six right angles, find the sum of angles 6, 7, 8, 9, and 10.



151. Summary. In this chapter you have learned

(1) The difference between theorems and problems.

(2) The meaning of *complementary*, *supplementary*, and *vertically opposite* angles.

(3) Theorems:

THEOREM 1. *Complements of the same or of equal angles are equal.*

THEOREM 2. *When one straight line meets another straight line, the sum of the adjacent angles is equal to two right angles.*

THEOREM 3. *Supplements of the same angle or of equal angles are equal.*

THEOREM 4. *If any number of straight lines meet at a point, the sum of all the adjacent angles is equal to four right angles.*

THEOREM 5. *If two straight lines intersect, the vertically opposite angles are equal.*

Test on Chapter IX

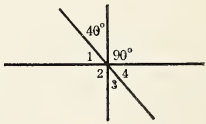
1. What are the two parts of a theorem?

2. What is meant by a problem?

3. Copy and complete the following table:

Angle	45°	32°	0°	50°	15°	90°
Complement . .						
Supplement . .						

4. What is the complement of an angle of $D°$? the supplement?



5. Find the sizes of the angles marked 1, 2, 3, and 4 in the diagram.

6. The interior angles of a triangle are 60°, 20°, 100° respectively. Find the size of each exterior angle.

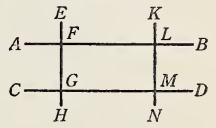
CHAPTER X

PARALLEL STRAIGHT LINES

152. Parallel Lines. *Two lines are said to be parallel when they lie in the same plane and do not cut each other.* By *line* we mean *a straight line* infinite in both directions. We sometimes speak of parallel lines as being the same distance (perpendicular distance) apart all the way along their length.

Exercises. *Parallel Straight Lines*

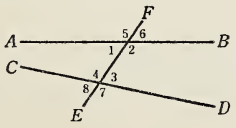
1. If AB and CD are parallel lines with EH and KN perpendicular to them, what lines and angles are equal?

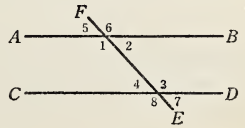


HINT. FG and LM each represent the distance between AB and CD.

2. $ABCD$ is a rectangle. Its opposite sides are equal; each angle is a right angle. Is $AB \parallel$ (parallel to) CD? $DA \parallel CB$?

153. Angles Made by a Transversal. EF, cutting AB and CD, is a *transversal*. $\measuredangle$ 1, 2, 3, 4 are *interior angles*. $\measuredangle$ 5, 6, 7, 8 are *exterior angles*. $\measuredangle$ 2 and 3 are interior angles on one side of the transversal. $\measuredangle$ 2 and 4 are *alternate angles*.





Exercises. *Angles Formed by a Transversal*

1. In Fig. 1 below, in which AB is not parallel to CD,

(1) Are the alternate angles equal? Test.

(2) Are the two interior angles on the same side of the transversal together equal to 180°?

2. Repeat Exercise 1 for Fig. 2 below, in which AB and CD are parallel.

THEOREM 6

When a straight line cuts two parallel straight lines, the alternate angles are equal.

THEOREM 7

When a straight line cuts two parallel straight lines, the interior angles on the same side of the cutting line are together equal to two right angles.

154. Corresponding Angles. Because they are in corresponding positions, the following pairs of angles are called *corresponding angles*: $\angle 6$ and $\angle 3$, $\angle 2$ and $\angle 7$, $\angle 5$ and $\angle 4$, $\angle 1$ and $\angle 8$.

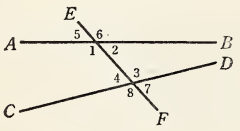


FIG. 1

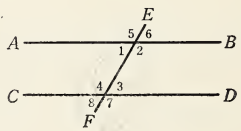


FIG. 2

We used the word *corresponding* in connection with congruent figures on page 242. When there is any doubt as to the meaning, it is better to be specific and say *corresponding angles of congruent figures* or *corresponding angles of parallel lines.*

Exercises. *Corresponding Angles*

1. In Fig. 1 on the opposite page, is $\angle 6 = \angle 3$? $\angle 2 = \angle 7$? $\angle 5 = \angle 4$? $\angle 1 = \angle 8$?

2. Repeat Exercise 1 for Fig. 2.

On the basis of the results obtained in these two exercises, we now accept as true the following theorem:

> THEOREM 8
>
> *When a straight line cuts two parallel straight lines, the corresponding angles are equal.*

155. Converse of a Theorem. In the statement or *enunciation* of a theorem, as was stated in section 145, there are two parts, first the hypothesis which is given (as true), and second the conclusion. For example, Theorem 6 may be divided as follows:

GIVEN: Two parallel straight lines.
CONCLUSION: The alternate angles formed when the lines are cut by a transversal are equal.

If we interchange the *conclusion* and the *given* part of the theorem, we have the *converse of the theorem.* We may state the converse of Theorem 6 as follows:

GIVEN: The alternate angles formed when two straight lines are cut by a transversal are equal.
CONCLUSION: The two lines are parallel.

Sometimes converse theorems are true; sometimes not. You must learn to think clearly to avoid drawing a false conclusion from any given hypothesis.

Exercises. *Testing Converses*

Read carefully each of the statements in Exercises 1 to 5, and decide whether its converse is true. First find the given part and the conclusion. Interchange these. Test the result.

1. If I am sick, I go to bed.
2. If a man is rich, he is happy.
3. Angles which are vertically opposite are equal.
4. A square is a figure which has four equal sides.
5. An angle of 90° is called a right angle.

156. Important Converses. Our experience with parallel lines convinces us that the converses of Theorems 6, 7, and 8 are true.

THEOREM 6*a*

When a straight line cuts two other straight lines, if a pair of alternate angles are equal, then the two straight lines are parallel.

THEOREM 7*a*

When a straight line cuts two other straight lines, if a pair of interior angles on the same side of the cutting line are together equal to two right angles, then the two straight lines are parallel.

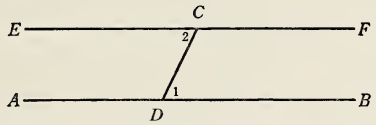
THEOREM 8*a*

When a straight line cuts two other straight lines, if a pair of corresponding angles are equal, then the two straight lines are parallel.

157. An Important Construction. Construction 6, which follows, will be found very useful.

CONSTRUCTION 6

Through a given point to construct a straight line parallel to a given straight line.



GIVEN: Line AB and any point C not in AB.

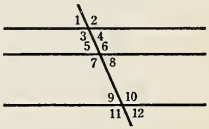
REQUIRED: To construct a straight line through C which is parallel to AB.

PLAN: Use transversal and equal alternate angles.

CONSTRUCTION: (Complete this in your notebook.)

Exercises. *Parallel Lines*

1. A transversal cuts three parallel straight lines as in the figure at the right.



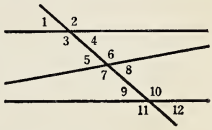
(1) Name the angles corresponding to $\angle 2$, $\angle 1$, $\angle 4$, $\angle 3$.

(2) Name all angles equal to $\angle 2$.

(3) Name all angles equal to $\angle 1$.

(4) Name sixteen pairs of angles whose sum is each 180°.

2. A transversal cuts three lines, as shown, only two of which are parallel.



(1) Name the angles corresponding to $\angle 2$; to $\angle 1$.

(2) Name all the angles that are equal to $\angle 2$; to $\angle 4$.

(3) Name all pairs of angles whose sum is 180°.

3. Given $AB = AC$ and $DE \parallel BC$ in Fig. 1 below. Do measurements indicate that $AD = AE$? that $DB = EC$?

4. Give in degrees the difference in direction between the lines AB and CD shown in Fig. 2 below.

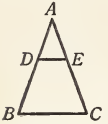


FIG. 1

FIG. 2

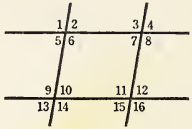
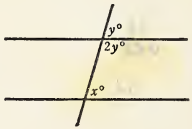


FIG. 3

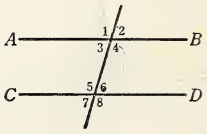
5. In Fig. 3 above, two parallels are cut by two other parallels. Name all the angles equal to $\angle 1$.

6. Two parallels are cut by a transversal forming the angles shown. Find the values of all angles shown in the figure.

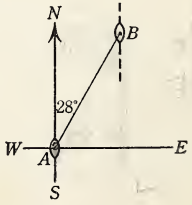


7. Are the lines AB and CD in the figure at the right always parallel when

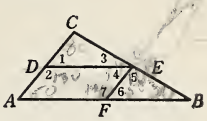
(1) $\angle 2 = 50°$, $\angle 6 = 50°$?
(2) $\angle 2 = 40°$, $\angle 3 = 40°$?
(3) $\angle 4 = 120°$, $\angle 6 = 80°$?
(4) $\angle 3 = \angle 5$?
(5) $\angle 4 = \angle 8$?
(6) $\angle 4 = \angle 6$?



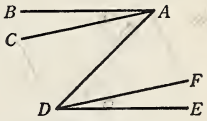
8. Parallel lines and angles are of great importance in navigation. The captain of a ship A says that the bearing of ship B is N. 28° E. Why? How would the captain of ship B describe the bearing of ship A? Which theorem applies here?



9. Given $\angle A = 50°$, $\angle B = 30°$, $\angle C = X°$, $DE \parallel AB$, and $EF \parallel CA$. What is the value of each of the angles shown?



10. Given $\angle CAD = \angle FDA$ and $\angle BAC = \angle EDF$ in the diagram at the right. Can you show that $AC \parallel FD$? that $AB \parallel DE$?



158. Summary. In this chapter you have learned

(1) The meaning of *parallel*.
(2) The meaning of *transversal*.
(3) The meaning of *interior*, *exterior*, and *corresponding* angles.
(4) The meaning of *converse* of a theorem.
(5) Theorems and Constructions:

THEOREM 6. *When a straight line cuts two parallel straight lines, the alternate angles are equal.*

THEOREM 6*a*. *When a straight line cuts two other straight lines, if a pair of alternate angles are equal, then the two straight lines are parallel.*

THEOREM 7. *When a straight line cuts two parallel straight lines, the interior angles on the same side of the cutting line are together equal to two right angles.*

THEOREM 7*a*. *When a straight line cuts two other straight lines, if a pair of interior angles on the same side of the cutting line are together equal to two right angles, then the two straight lines are parallel.*

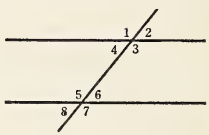
THEOREM 8. *When a straight line cuts two parallel straight lines, the corresponding angles are equal.*

THEOREM 8*a*. *When a straight line cuts two other straight lines, if a pair of corresponding angles are equal, then the two straight lines are parallel.*

CONSTRUCTION 6. *Through a given point to construct a straight line parallel to a given straight line.*

Test on Chapter X

1. In this figure name (1) four interior angles; (2) four exterior angles; (3) four pairs of corresponding angles; (4) four pairs of vertically opposite angles.



2. Complete the following statements:

(1) When a straight line cuts two parallel straight lines, the _(?)_ angles and also the _(?)_ angles are equal.

(2) When a straight line cuts two parallel straight lines, the interior angles on the same side of the cutting line are _(?)_.

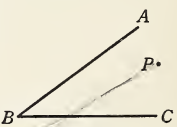
(3) When the _(?)_ and the _(?)_ of a theorem are interchanged, we have the converse of the theorem.

(4) The converse of a theorem is _(?)_ true.

(5) When a straight line cuts two other straight lines so that a pair of _(?)_ angles or a pair of _(?)_ angles are equal, then the two straight lines are parallel.

(6) When a straight line cuts two other straight lines, if a pair of interior angles on the same side of the cutting line are _(?)_, then the two straight lines are parallel.

3. Draw a diagram similar to the one shown here, but larger. Through P construct a line parallel to AB; to BC.



4. State the converse of each of the following statements, indicating whether or not it is true:

(1) If two straight lines are cut by a transversal so that the alternate angles are equal, then the lines are parallel.

(2) Triangles which have corresponding sides equal have corresponding angles equal.

CHAPTER XI

ANGLES OF A TRIANGLE; FORMAL PROOFS

159. How to Prove Theorems and Solve Problems. You are now ready to begin the work of proving statements in geometry. As you read the following instructions, refer to the proof of Theorem 9 on page 265.

1. Read the given theorem or problem. Read it repeatedly until you are certain you have the meaning of each word.

2. Draw a freehand sketch showing the conditions given. Study it while deciding upon a plan of attack. When you have decided on a plan, draw an accurate figure.

3. State what is *Given*.

4. State what is *Required*.

5. State your plan of procedure.

6. Describe the changes necessary in the figure, if any. As these are described, make the constructions.

7. Write the proof. In a *Theorem* the proof is the argument that shows that the conclusion follows logically from the hypothesis and the preceding propositions, etc. In a *Problem* the proof is the argument that shows your construction to be correct. Each step of the argument must be supported by evidence, as explained in the following section.

160. What shall we Accept as a Reason in a Proof? The term *formal proof* appears in the chapter title. What does this mean? Have we proved any statements previously?

Up to this point we have drawn, observed, and measured many lines, angles, and figures, and our experience has led

us to say that certain statements *seem true.* We have been willing to go ahead on the assumption that they are facts. For example, we assumed that "vertically opposite angles are equal." It appears obvious. We might have called this statement an *assumption* instead of Theorem 5. All the propositions thus far which we have called theorems are really assumptions, because formal proofs were not given.

A formal proof is one, such as for Theorem 9, where the argument is presented step by step, with sequence correct, and with all reasons formally stated. Recognized assumptions are given as reasons. Later we shall also use theorems which have been formally proved.

161. Additional Assumptions. The following additional assumptions will be accepted and used as required:

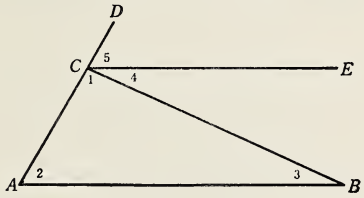
1. *Things which are equal to the same thing are equal to one another.*
2. *If equals are added to equals, the sums are equal.*
3. *If equals are subtracted from equals, the remainders are equal.*
4. *If equals are multiplied by equals, the products are equal.*
5. *If equals are divided by equals, the quotients are equal.*
6. *The whole is greater than its part and equal to the sum of all its parts.*
7. *If equals are added to unequals, the sums are unequal.*
8. *If equals are subtracted from unequals, the remainders are unequal.*
9. *If the first of three quantities is greater than the second and the second is greater than the third, then the first is greater than the third.*

You have already studied some of these assumptions on page 54.

162. One Fact True for All Triangles. In geometry you have met many kinds of triangles. Theorem 9, which follows, is true for all triangles.

THEOREM 9

The sum of the angles of a triangle is equal to two right angles.



GIVEN: $\triangle ABC$.

REQUIRED: To prove $\angle 1 + \angle 2 + \angle 3 = 180°$.

PLAN: Use CB as a transversal with two parallel lines.

CONSTRUCTION: Produce AC to D. Draw $CE \parallel AB$.

PROOF:

STATEMENTS	REASONS
1. Consider CB as a transversal cutting the parallel lines AB and CE.	
2. $\angle 4 = \angle 3$.	2. Alternate angles.
3. $\angle 5 = \angle 2$.	3. Corresponding angles.
4. $\angle 4 + \angle 5 = \angle 3 + \angle 2$.	4. Equals added to equals.
5. Add $\angle 1$ to each side.	
6. $\angle 1 + \angle 4 + \angle 5 = \angle 1 + \angle 3 + \angle 2$.	6. Equals added to equals.
7. $\angle 1 + \angle 4 + \angle 5 = 180°$.	7. A straight angle.
8. $\angle 1 + \angle 2 + \angle 3 = 180°$.	8. Things equal to the same thing are equal.

As an exercise rewrite this theorem, naming the angles by letters.

163. Relation of Exterior and Interior Angles of a Triangle. In Fig. 1, we know that ∡ 1 and 2 are called interior angles and ∡ 3 and 4 are called exterior angles.

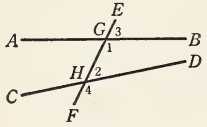


FIG. 1

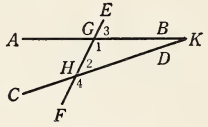
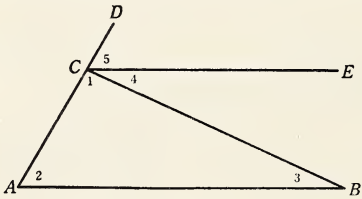


FIG. 2

Supposing now that AB and CD are produced so that they meet at K as in Fig. 2. We now have a $\triangle KGH$, and ∡ 1 and 2 are said to be interior angles of this triangle while ∡ 3 and 4 are said to be exterior angles of the triangle.

THEOREM 10

If one side of a triangle is produced, the exterior angle so formed is equal to the sum of the two opposite interior angles.



GIVEN: $\triangle ABC$ with side AC produced to D.

REQUIRED: To prove $\angle BCD = \angle A + \angle B$.

PLAN: Add construction lines to make BC a transversal. Make use of the properties of parallel lines and transversals.

CONSTRUCTION: From C draw $CE \parallel AB$.

PROOF:

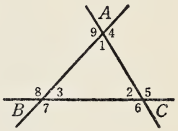
STATEMENTS	REASONS
(Complete proof: Compare $\angle 4$ with $\angle 3$; compare $\angle 5$ with $\angle 2$. Use assumption 2, section 161.)	

NOTE. This theorem was really proved in the course of Theorem 9, but it is given again here partly for practice and partly because it states a most important property of triangles which you will frequently make use of.

QUIZ: 1. Show that there are six exterior angles for any triangle.

2. In the figure at the right, name the exterior angle that is equal to $\angle 1 + \angle 2$.

3. What construction would be necessary to prove this?



164. Formula for Theorem 10. The truth expressed in Theorem 10 may be given in a formula.

If A, B, and C represent the number of degrees in the three angles of a triangle, we have

$$A + B + C = 180°,$$

by Theorem 9.

Subtracting A from each side, the equation becomes

$$B + C = 180° - A,$$

by assumption 3, page 264, or

$$180° - A = B + C.$$

Note that $180° - A$ is the number of degrees in the supplement of the angle A, that is, the exterior angle.

QUIZ: 1. What is the formula for the exterior angle at B?

2. What is the formula for the exterior angle at C?

165. Useful Information regarding Exercise-Solving. Exercises in geometry are interesting; they are like little puzzles. Here are a few suggestions that you can use when solving:

1. Draw accurate figures. Accurate figures help greatly by revealing angles that are equal, lines that are equal, lines that are parallel, and other relationships.

2. Do not put restrictions on figures when these are not necessary. For example, make it your practice to draw a scalene triangle if you are to draw a triangle that is not named as isosceles or equilateral.

If two isosceles triangles are to be drawn on the same base and on opposite sides of it, draw triangles as in Fig. 1, not as in Fig. 2. Why?

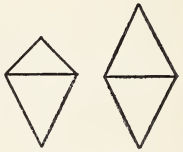


FIG. 1 FIG. 2

If a transversal cuts two straight lines that are not specified as parallel, draw them as in Fig. 3, not as in Fig. 4 or Fig. 5.

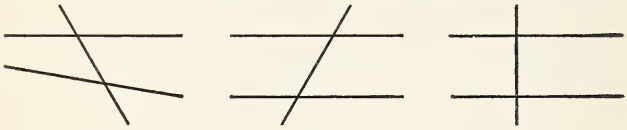
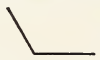
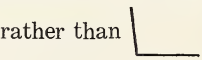
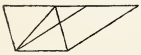


FIG. 3 FIG. 4 FIG. 5

3. Make special features of a figure very prominent. For an obtuse angle draw  rather than 

4. If the construction or the figure has many parts, it will help you if a few parts are drawn in light lines, others in heavy lines. This figure is preferable to  . Different parts may be drawn in different colors.

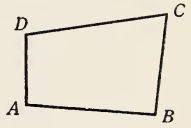
5. Sometimes it helps to shade part of the figure. Two congruent triangles if buried in one figure may stand out clearly if one of them is shaded or lightly crosslined.

Exercises. *Angles of Plane Figures*

Use the suggestions just given to help you solve the exercises which follow. When an exercise requires a proof, follow carefully the form shown for Theorem 9 on page 265.

1. GIVEN: $ABCD$ is a quadrilateral.

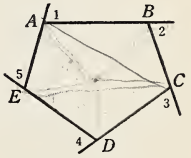
TO PROVE: $\angle A + \angle B + \angle C + \angle D = 360°$.



PLAN: Divide the quadrilateral into triangles.

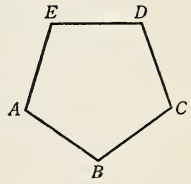
2. GIVEN: Polygon $ABCDE$.

TO PROVE: $\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5 = 360°$.



PLAN: Divide the polygon into triangles. Find the sum of the interior angles of the polygon. Use the fact that there are straight angles at A, B, C, D, and E.

3. The figure at the right is a *regular pentagon*. It has 5 equal sides and 5 equal angles.



(1) Calculate the size of each angle.

(2) What is the sum of the five exterior angles? Note that these five angles are equal.

4. If two angles of one triangle are equal to two angles of another triangle, the third angles are also equal. Write in your notebook the complete proof of this proposition.

5. One angle of a triangle is equal to the sum of the two other angles. What special kind of triangle is this? How many such triangles can be drawn no two of them being of the same size?

6. One angle of a triangle is $X°$. Another angle of the same triangle is 90°. Calculate the third angle.

7. Copy and fill in the blanks:

ANGLES OF A TRIANGLE

First Angle	Second Angle	Third Angle	Check
90°	40°	_(?)_	_(?)_
90°	60°	_(?)_	_(?)_
90°	$X°$	_(?)_	_(?)_
$X°$	$Y°$	_(?)_	_(?)_

8. If the angle between the equal sides of an isosceles triangle is 48°, what is the size of the exterior angle at the base?

9. In a triangle ABC the angle A contains 48 degrees. Angles B and C are bisected by straight lines which meet in O. Deduce the number of degrees in the angle BOC.

10. How many degrees are there in each angle of an isosceles right-angled triangle?

11. If the three expressions $2X$, $3X - 4$, and $X - 8$ represent the number of degrees in the three angles of a triangle, find the number of degrees in each angle.

12. Find the number of degrees in each angle of a triangle if one angle is twice the second and three times the third.

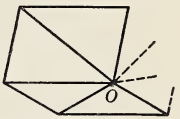
13. Explain why a triangle can have no more than one right angle.

14. How many degrees are there in each exterior angle of an equilateral triangle?

15. How large is each angle in a regular hexagon?

NOTE. A regular hexagon has six equal sides and six equal angles.

16. Find the sum of the interior angles of a polygon with N (an indefinite number of) sides.



SUGGESTIONS. 1. Join all vertices to any point O within the polygon.

2. How many triangles will be formed if the polygon has N sides?

3. Sum of angles at $O =$ _(?)_.

4. Complete the proof that the sum of the interior angles is $(2N - 4)$ right angles.

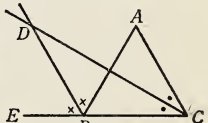
17. Copy the following table in your notebook and fill in the missing figures:

REGULAR POLYGON OF N SIDES	EACH INTERIOR ANGLE IS
$N = 5$	_(?)_
$N = 6$	_(?)_
$N = 18$	_(?)_
_(?)_	160°
_(?)_	108°
_(?)_	135°
_(?)_	144°
$N = 10$	_(?)_

18. Show that if the sides of a polygon are produced in both directions, sets of exterior angles are formed whose sum is 8 right angles.

19. GIVEN: $\triangle ABC$ with (1) interior angle at C bisected by CD; (2) exterior angle at B bisected by BD.

TO PROVE: $\angle A = 2 \angle D$.



SUGGESTIONS. 1. Give algebraic values to angles at A, B, and C.

2. Find value of $\angle DBC$, which is equal to $\angle B + \frac{1}{2} \angle ABE$.

3. Use fact that $\angle A + \angle B + \angle C = 180°$.

166. Summary. In this chapter you have learned

(1) How to prove theorems and solve problems.
(2) The meaning of *assumptions*.
(3) Nine important assumptions often used in proofs.
(4) Theorems:

THEOREM 9. *The sum of the angles of a triangle is equal to two right angles.*

THEOREM 10. *If one side of a triangle is produced, the exterior angle so formed is equal to the sum of the two opposite interior angles.*

Test on Chapter XI

1. State the seven steps which should be taken in proving a theorem or solving a problem.

2. State nine common assumptions learned in this chapter.

3. What is the sum of the angles of a triangle? Is your answer true for all triangles?

4. If the angles of a triangle are X, Y, and Z, write the formulas for the sizes of the exterior angles.

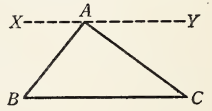
5. Is it possible to have two right angles in a triangle?

6. Can an isosceles triangle contain a right angle? Why?

7. The angles of a triangle are in the ratio of 5, 6, and 7. Find the size of each angle.

8. In a $\triangle ABC$, $\measuredangle A$ and B are complementary. Prove that $\angle C$ is a right angle.

9. Prove Theorem 9 by making use of the diagram at the right and without drawing any further construction lines.

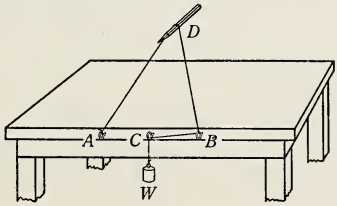


CHAPTER XII

CONGRUENCE

1. Simple Apparatus

167. The String Triangle. Insert three screw eyes just below the rear edge of a table. Tie the end of a strong string at A. Thread the other end through B, and, bringing it back through C, tie a weight W to the end. By inserting a pencil under the string between A and B, the string may be lifted to form a triangle. D may be moved about into various positions to illustrate many points regarding the angles, the relation of sides, and the area of triangles.



1. Consider changes in the triangle as D travels vertically up and down directly above C, the middle point of AB.

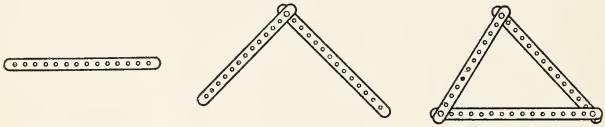
2. Consider changes in $\measuredangle A$, B, and D as the vertex D moves to the right or to the left.

3. Consider areas of various triangles that may be formed on the base AB.

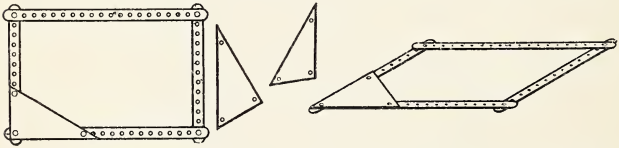
4. Use the apparatus to show the *continuity* of changing relationships. The acute angle changes through a right

angle to an obtuse angle; the isosceles triangle changes altitude *gradually*, the vertex passes vertically through one point such that, for a moment, an equilateral triangle is formed, and then once more there is an isosceles triangle.

168. Making Models. At intervals of one-half inch punch holes through narrow strips, of various lengths, made of wood or strong cardboard. Use small bolts and nuts or paper-fasteners to hold the strips securely in any position desired after two or more strips have been joined together. Have as many as five identical strips of selected lengths.



Make several triangular pieces with angles of various sizes. If these triangles are perforated near the vertices, they may be used with the strips to study many plane figures.



1. Consider figures when a right-angled triangle is used with four strips of different lengths.

2. Consider changes in a figure when an angle is kept constant and lengths of sides vary in different ways.

3. Repeat experiment 2, using angles of different sizes.

4. Study the number of sides or angles in different plane figures that must be fixed before the structure becomes rigid. Test conditions of congruence.

2. Congruent Triangles

169. Constructing Congruent Triangles. As you already know, if two triangles are so related that their corresponding sides are equal and their corresponding angles are equal, the triangles are congruent. In the following exercises we shall make some experiments which will lead us to see the truth of three important theorems.

Exercises. *Experimenting with Triangles*

1. (1) Use the model apparatus described on the opposite page to study the effect of keeping constant the magnitude of two sides and a contained angle of a triangle.

(2) Assign lengths for the two sides and name a definite angle measure for the contained angle. Let several pupils cut cardboard triangles meeting these specifications. Compare all the triangles.

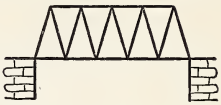
In this exercise we have been concerned with triangles in which *two sides and the contained angle are of constant size.*

2. Make experiments similar to those of Exercise 1, using triangles in which *two angles and the included side are of constant size.*

3. Make experiments similar to those of Exercise 2 but using triangles which have *any* two angles and a side of constant size. What do you conclude? Would Theorem 9 in conjunction with Exercise 2 lead you to the same conclusion?

4. Again make experiments similar to those of Exercise 1, this time using triangles in which *the three sides are of constant lengths.*

5. Here is a diagram showing a common type of steel bridge. Why do you think engineers build bridges using triangles instead of figures with four or five sides?



170. Three Theorems. As a result of earlier construction work (section 142), together with the experiments just performed, we may now accept without proof the following theorems:

THEOREM 11 (S.A.S.)*

If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are congruent.

THEOREM 12 (A.S.A.)

If two triangles have two angles and a side of one equal respectively to two angles and the corresponding side of the other, the triangles are congruent.

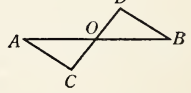
THEOREM 13 (S.S.S.)

If two triangles have the three sides of one equal respectively to the three sides of the other, the triangles are congruent.

Exercises. *Congruent Triangles*

1. GIVEN: AB and CD are straight lines, $AO = OB$, and $CO = OD$.

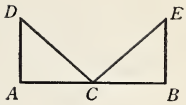
TO PROVE: $\triangle AOC$, ODB are congruent.



* The abbreviations shown serve as a convenient way of referring to these theorems. Thus, s.a.s. = side-angle-side.

2. GIVEN: AD and BE are $\perp AB$, C bisects AB, and $AD = BE$.

TO PROVE: $\triangle ADC$ and BEC are congruent.

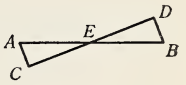


3. GIVEN: BC bisects $\angle ABD$, and $BA = BD$.

TO PROVE: $\angle A = \angle D$.

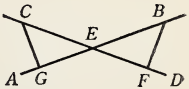
4. GIVEN: AB and CD are straight lines, E is the midpoint of CD, and AC and DB are $\perp CD$.

TO PROVE: $\triangle ACE$ and BDE are congruent.



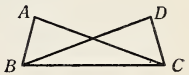
5. GIVEN: AB and CD are straight lines,
$CG \perp AB$,
$BF \perp CD$,
$CG = BF$.

TO PROVE: (1) $\angle C = \angle B$, (2) $GE = FE$.



6. GIVEN: $AB = DC$,
$AC = DB$.

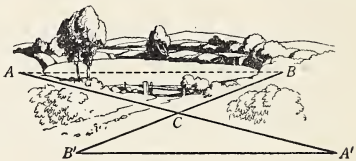
TO PROVE: $\angle A = \angle D$.

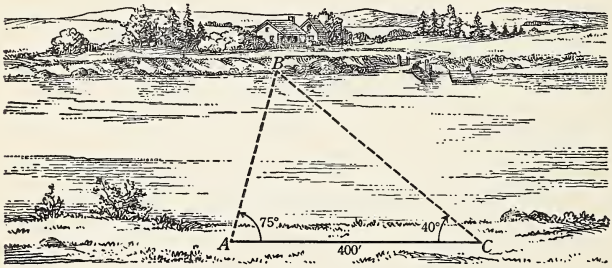


Exercises. *Measuring Distances Indirectly*

1. Suppose that a surveyor wishes to measure the distance between points A and B on the ground, but that because of some obstacle, such as a pond, he cannot measure it directly. The diagram suggests one way in which he can find this distance indirectly. Can you explain it?

2. A second way of measuring the distance AB when A and B are so located that the direct measurement of AB is impossible is suggested by the figure below. From a convenient point C measure the length of CA and CB. Extend AC and BC beyond C, making $CA' = CA$ and $CB' = CB$. Then $\triangle A'B'C$ is congruent to $\triangle ABC$. (Why?) Hence $A'B'$ is equal to AB. That is, by measuring $A'B'$ *directly*, we may determine the length of AB *indirectly*. Draw a diagram illustrating the method shown above and explain in your own words how the length of AB is measured indirectly.

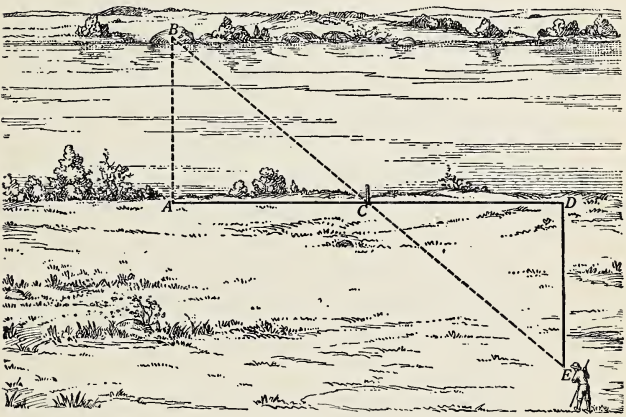




3. In the diagram above, A and B represent two points on opposite banks of a river. If $AC = 400$ feet, $\angle A = 75°$, and $\angle C = 40°$, find the length of AB by making a scale drawing of $\triangle ABC$.

4. An interesting story is told of one of Napoleon's men. The famous general wished to measure the distance across a river, but no measuring instruments were available. How-

ever, one of his soldiers stood on the bank of the river at a point P and sighted a point Q on the opposite bank so that it was in a direct line with the peak of his cap. Then without raising or lowering his head, he turned around and sighted a point R on the ground. He then measured the distance PR and said that it was equal to the width of the river. The story has it that this clever man was immediately promoted by Napoleon. Can you show that the man's method was correct?



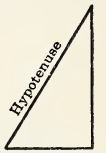
5. The boy in the diagram above is using an ancient method of finding the width (AB) of a stream. A distance AC is laid off along the bank at right angles to AB. At C a vertical rod is set up. The line AC is extended to D, making CD the same length as AC. Then DE is laid off at right angles to AD. The boy walks along DE until he finds a point E in a straight line with C and B. Which distance must be measured to find the width of the river? Prove.

171. Special Congruence Theorem for Right-Angled Triangles. In the case of right-angled triangles (and right-angled triangles only) there is a special extension of the congruence theorems.

> THEOREM 14
>
> *If two right-angled triangles have the hypotenuse and one side of one equal to the hypotenuse and one side of the other, the triangles are congruent.*

QUIZ: Why can the word "respectively" be omitted in the enunciation of Theorem 14?

NOTE. In a right-angled triangle the side which is opposite to the right angle is called the hypotenuse.



Right triangle

172. Summary of Congruence Theorems. At this point you will find it helpful to review the facts necessary for proving two triangles congruent. Be sure that you understand the following statement:

Triangles are congruent if they have the following *corresponding* parts equal:

(1) *three* sides.
(2) *two* sides and *one* angle (included).
(3) *one* side and *two* angles.

QUIZ: 1. Do you think it important for the equal parts to be *corresponding* parts also?

2. Note the effect in each congruence theorem when the equal parts are *not* corresponding parts.

3. Draw diagrams to illustrate your conclusions.

4. Are triangles congruent when three corresponding angles are equal? Draw diagrams to illustrate your answer.

173. Practice in Naming Corresponding Parts of Triangles. A useful device for finding equal sides or angles of congruent triangles is to draw arrows from the known equal sides to the angles opposite them, or from known equal angles to the sides opposite. Thus, if AB is known to equal DE, draw arrows as shown in Fig. 1. They lead to the equal angles C and F. If $\angle K$ is known to be equal to $\angle M$, draw arrows as shown in Fig. 2. They lead to the equal sides LH and ON.

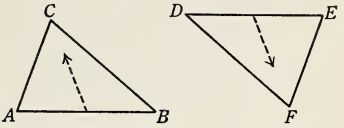


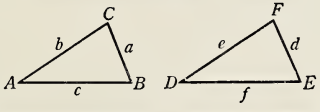
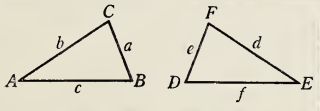
FIG. 1

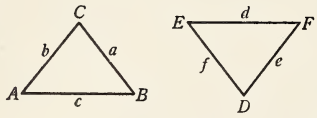
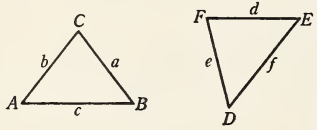
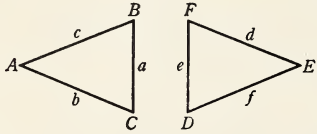
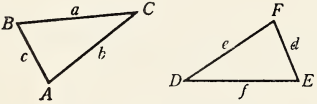
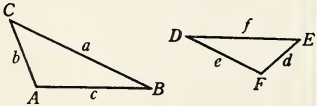
FIG. 2

The following exercises will help you to distinguish the corresponding parts of given triangles.

Exercises. *Finding Corresponding Parts*

Copy and complete the table below and on page 282:

	FIGURES	GIVEN	CORRESPONDING PARTS
1.		$b = e$ $A = D$ $c = f$	*Solution.* A and D. a and d. B and E. b and e. C and F. c and f.
2.		$c = f$ $B = D$ $a = e$	

	Figures	Given	Corresponding Parts
3.		$c = d$ $a = f$ $B = E$	
4.		$A = E$ $b = d$ $c = f$	
5.		$c = f$ $b = d$ $a = e$	
6.		$a = f$ $B = E$ $C = D$	
7.		$c = f$ $A = F$ $B = D$	(Note that the equal sides here do not correspond. Are the triangles congruent?

174. Assumptions and Theorems. In section 160 we discussed the assumptions which had been made up to that point. In sections 161–173 we have made further assumptions. Review these sections and make a list of all the new assumptions.

QUIZ: 1. How many of these are stated in the text to be assumptions?

2. Do they appear to be reasonable assumptions?

A good way to test many of these assumptions is to try them out with simple figures. For example, No. 2, section 161: Let us say 5 yards = 15 feet. Add an equal length, say 2 feet, to both sides. Is it true that 5 yards 2 feet = 17 feet?

You have doubtless observed that some of the theorems studied thus far have been proved and that others have not; that is, the latter are assumed to be true. The proof of a theorem demonstrates that the conclusion is true if the assumptions and theorems used in the course of the proof are true. In geometry we must always make a set of basic, or fundamental, assumptions. The theorems which we prove are conclusions which depend upon the truth of these initial assumptions.

In different textbooks different initial assumptions are made, and so it frequently happens that different propositions are proved. Even with the same set of basic assumptions, the order in which the theorems are proved is likely to be different in different texts; the order in any given text is only one of the many ways in which a logical geometrical course may be built. You should therefore satisfy yourself that you understand each proposition not only by itself but also in its relation to other propositions.

As you study each proof, guard against merely memorizing without understanding.

175. Equalities in an Isosceles Triangle. Cut from paper an isosceles triangle. Fold the triangle so that the base overlaps upon itself with its ends together. If ABC (Fig. 1) represents the triangle, then when folded the paper will appear as in Fig. 2 with A and B coinciding, and the crease running from the middle point of the base to the vertex C.

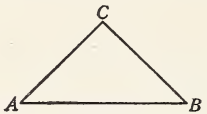


FIG. 1

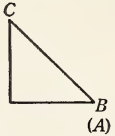


FIG. 2

Evidently CA and CB coincide, since $CA = CB$. The $\measuredangle$ CAB and CBA now coincide, since the vertex of each is at B and their arms coincide throughout their lengths. This means that $\angle CAB = \angle CBA$.

To our list of assumptions we may add the following:

THEOREM 15

The angles at the base of an isosceles triangle are equal.

THEOREM 16

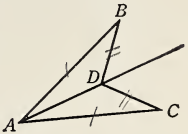
If the angles at the base of a triangle are equal, the sides opposite to these angles are equal.

Note that Theorem 16 is the converse of Theorem 15. Re-read section 155.

As an exercise you may wish to prove Propositions 15 and 16 instead of leaving them as assumptions. Begin by drawing the line that would bisect the vertex angle. Test the two triangles formed for congruence.

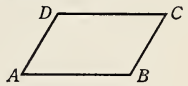
Exercises. *Using the Congruence Theorems*

1. GIVEN: $AB = AC$, $BD = CD$.
TO PROVE: $\triangle ABD \equiv \triangle ACD$.

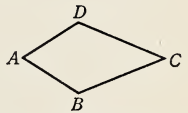


NOTE. $\equiv$ means *congruent to*.

2. GIVEN: $DC = AB$, $AD = BC$.
TO PROVE: $\angle BAD = \angle BCD$.



3. GIVEN: $AB = AD$, $BC = DC$.
TO PROVE: $\angle B = \angle D$.



4. GIVEN: A straight line PQ and a point R which is not on this line but is equidistant from P and Q.
TO PROVE: The line joining R to the mid-point of PQ is perpendicular to PQ.

176. Formulas Connecting Angles in an Isosceles Triangle. If V is the number of degrees in the vertex angle of an isosceles triangle and B is the number of degrees in an angle at the base, then

(1) $V + 2B = 180°$.

(2) $V = 180° - 2B$.

(3) $B = \frac{1}{2}(180° - V)$.

Exercises. *Isosceles Triangles*

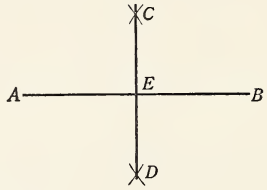
1. Use formula (2) to find the vertex angle if each base angle is 72°.

2. Use formula (3) to find a base angle if the vertex angle is 45°.

177. Using a Congruence Theorem to Verify the Method of Bisecting a Given Straight Line. You can now prove the accuracy of your usual method of bisecting a straight line.

CONSTRUCTION 1

To bisect a given straight line.



GIVEN: Line AB.

REQUIRED: To find a point E, in AB, such that $AE = EB$.

PLAN: Arrange the construction so that AE and EB will be corresponding sides of congruent triangles.

CONSTRUCTION: 1. With centres A and B and equal radii greater than $\frac{1}{2}\,AB$ construct two arcs intersecting at C. (Why must radii be greater than $\frac{1}{2}\,AB$?)

2. Let these two arcs intersect also at D on the other side of AB.

3. Join CD.

PROOF:

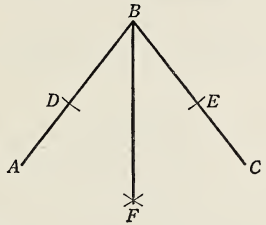
STATEMENTS	REASONS

Complete $\triangle ACD$, BCD. Use s.s.s. theorem. You will be able to prove $\angle ACE = \angle BCE$. Then compare $\triangle ACE$ and BCE. This time use s.a.s. theorem to prove $AE = EB$. Complete all statements and reasons in your notebook.

178. Using a Congruence Theorem to Verify the Method of Bisecting a Given Angle. You can now prove the accuracy of the construction you learned in section 133.

CONSTRUCTION 2

To bisect a given angle.



GIVEN: $\angle ABC$.

REQUIRED: To bisect $\angle ABC$.

PLAN: Construct a figure to which the s.s.s. theorem will apply.

CONSTRUCTION: 1. With centre B and any radius draw an arc cutting BA and BC at D and E respectively.

2. With centres D, E and equal radii construct two arcs intersecting at F.

What is the shortest radius that can be used for this second circle?
What is the greatest length this radius could have?
Must the radius BD be equal to the radius DF?

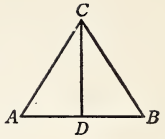
3. Draw BF.

PROOF:

STATEMENTS	REASONS

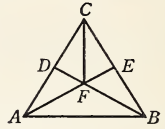
Write the proof in your notebook. Complete $\triangle BDF$, BEF. Use s.s.s. theorem.

Exercises. *Review*



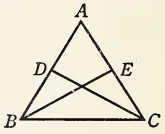
1. GIVEN: ABC is an isosceles triangle, with $CA = CB$ and $AD = BD$.

   TO PROVE: $CD \perp AB$.



2. GIVEN: $CA = CB$,
   $CD = CE$.

   TO PROVE: (1) CF bisects $\angle DCE$,
   (2) CF produced bisects AB.



3. GIVEN: $AB = AC$,
   $AD = DB$,
   $AE = EC$.

   TO PROVE: $BE = CD$.

4. ABC is a given angle. Divide it into four equal parts.

5. If a triangle is equiangular, it must also be equilateral.

6. If two straight lines bisect each other, the lines joining their extremities are equal in pairs.

7. The line bisecting the vertex angle of an isosceles triangle is perpendicular to the base and bisects it.

8. If one angle of a parallelogram is a right angle, the figure is a rectangle.

9. If two triangles are congruent, their corresponding medians are equal.

NOTE. A *median* of a triangle is the line drawn from any vertex to the mid-point of the opposite side.

10. If the line drawn from the vertex of a triangle to the middle point of the base is perpendicular to the base, the triangle is isosceles.

11. The diagonals of a quadrilateral bisect each other at right angles. Prove that the quadrilateral is a rhombus, that is, a parallelogram with equal sides.

12. The diagonals of a rhombus $ABCD$ bisect the angles A, B, C, and D.

13. Construct, with ruler and compasses only, an angle of 60°.

14. Construct, with ruler and compasses only, an angle of 30°.

179. **Summary.** In this chapter you have learned

(1) More about congruence.

(2) How to make simple geometric models.

(3) Theorems:

THEOREM 11. *If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are congruent.*

THEOREM 12. *If two triangles have two angles and a side of one equal respectively to two angles and the corresponding side of the other, the triangles are congruent.*

THEOREM 13. *If two triangles have the three sides of one equal respectively to the three sides of the other, the triangles are congruent.*

THEOREM 14. *If two right-angled triangles have the hypotenuse and one side of one equal to the hypotenuse and one side of the other, the triangles are congruent.*

THEOREM 15. *The angles at the base of an isosceles triangle are equal.*

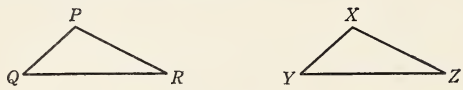
THEOREM 16. *If the angles at the base of a triangle are equal, the sides opposite to these angles are equal.*

(4) How to measure distances between inaccessible points.

(5) To use the theorems so far studied to verify the method of bisecting straight lines and angles.

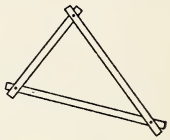
Test on Chapter XII

1. $\triangle PQR$ and XYZ are congruent. Name, in pairs, the corresponding sides and angles.



2. State four sets of conditions under which two triangles are congruent.

3. Three rods are hinged at the ends as shown in the diagram. Can you make a different triangle by joining these rods in a different order? Give reasons for your answer.



4. If four rods are hinged to form a quadrilateral, is a rigid figure formed? How could a fifth rod be used to insure rigidity? Prove that your answer is correct.

5. A boat is anchored in a lake and Tom wishes to find how far it is from the shore. How can he determine this distance without leaving land?

6. What properties are common to all isosceles triangles?

7. What formulas connect the angles of an isosceles triangle?

8. Prove that the base angles of an isosceles triangle are equal.

9. State and prove the converse of the theorem stated in Exercise 8.

10. State and prove the s.s.s. theorem.

11. Draw any angle PQR. Bisect the angle and prove that your construction is correct.

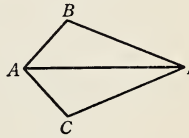
CHAPTER XIII

PRACTICE ON ESSENTIALS

180. Selecting the Correct Congruence Theorem for a Given Exercise. It is important to be able to recognize the particular congruence theorem which fits a given situation. These exercises will help to develop your skill in this respect.

Exercises. *Using Congruence Theorems*

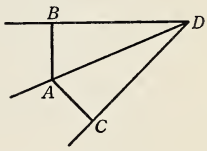
In each of the following exercises name (1) *the corresponding equal parts of the given triangles;* (2) *the congruence theorem to be used in the proof* (*Theorem 11, s.a.s.; Theorem 12, a.s.a.; Theorem 13, s.s.s.*). *Then write the proof in full in your notebook.*



1. In the quadrilateral $ABDC$, $AB = AC$ and $BD = CD$. Prove that $\angle BDC$ is bisected by AD.

(1) _(?)_, _(?)_, and _(?)_ are respectively equal to _(?)_, _(?)_, and _(?)_.

(2) Theorem _(?)_ is used.

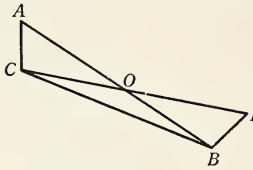


2. DA bisects $\angle BDC$, and AB and AC are $\perp DB$ and DC respectively. Prove that $AB = AC$.

(1) _(?)_, _(?)_, and _(?)_ are respectively equal to _(?)_, _(?)_, and _(?)_.

(2) Theorem _(?)_ is used.

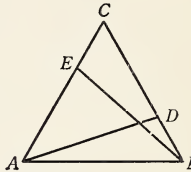
3. In $\triangle ABC$ and DCB, $AC = DB$ and $AB = DC$. Prove that $\angle OCB = \angle OBC$.



(1) _(?)_, _(?)_, and _(?)_ are respectively equal to _(?)_, _(?)_, and _(?)_.

(2) Theorem _(?)_ is used.

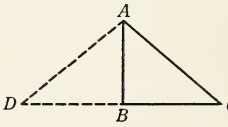
4. ABC is an equilateral triangle, and $BD = CE$. Prove that $\angle AEB = \angle ADC$.



(1) _(?)_, _(?)_, and _(?)_ are respectively equal to _(?)_, _(?)_, and _(?)_.

(2) Theorem _(?)_ is used.

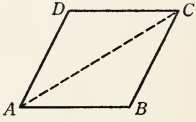
5. $\angle B$ of $\triangle ABC$ is a right angle, and CB is produced to D, making $BD = BC$. Prove that $AD = AC$.



(1) _(?)_, _(?)_, _(?)_ are respectively equal to _(?)_, _(?)_, _(?)_.

(2) Theorem _(?)_ is used.

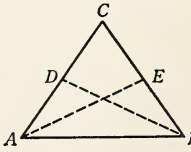
6. $ABCD$ is a rhombus. AC is a diagonal. Prove that the angles at A and C are bisected by AC.



(1) _(?)_, _(?)_, and _(?)_ are respectively equal to _(?)_, _(?)_, and _(?)_.

(2) Theorem _(?)_ is used.

7. In $\triangle ABC$, $AC = BC$. D and E are the middle points of AC and BC respectively. Prove that $AE = BD$.



(1) _(?)_, _(?)_, and _(?)_ are respectively equal to _(?)_, _(?)_, and _(?)_.

(2) Theorem _(?)_ is used.

181. Practice in Drawing Figures for Given Exercises. After reading an exercise it is often necessary to draw an accurate figure. The following exercises are designed to give practice in this important step of the solution.

Exercises. *Drawing Accurate Figures*

Draw the figures described below. No explanations are required.

1. Two straight lines AOB and COD cut one another at O, so that $OA = OB$ and $OC = OD$. Join AD and BC.

2. ABC is an isosceles triangle on the base AB. AC is produced to D, so that $AC = CD$.

3. The perpendicular bisector of the side AB of the triangle ABC intersects AC at D and BC produced at E.

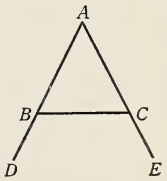
4. PQR is a triangle having $\angle PQR = \angle PRQ$. QS is a line bisecting $\angle PQR$ and meeting PR at S. RT is a line bisecting $\angle PRQ$ and meeting PQ at T. ST is joined.

182. Profitable Questions to Ask when Trying to Prove Two Angles Equal or Find Two Angles that are Equal. In proving theorems and exercises it is often necessary, in the course of the proof, to show that two angles are equal or to find two angles that are equal. You will find it helpful to ask yourself the following questions:

1. Are they complements or supplements of the same or equal angles?

Example. GIVEN: ABC is an isosceles triangle, AB and AC are produced to D and E.

TO PROVE: $\angle DBC = \angle ECB$.



2. Are they the sum or difference of two equal pairs of angles?

Example. GIVEN: $PQ = AB$,
$PR = AC$,
$\angle QPS = \angle BAD$,
$\angle SPR = \angle DAC$.

TO PROVE, first for Fig. 1 and then for Fig. 2: $\triangle PQR \equiv \triangle ABC$.

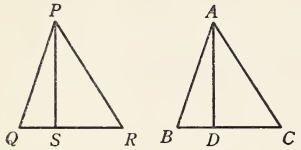


FIG. 1

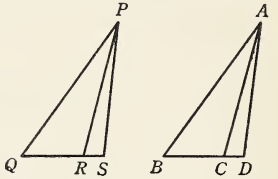
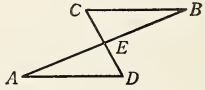


FIG. 2

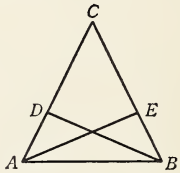
3. Are they vertically opposite?

Example. GIVEN: $AE = EB$,
$CE = ED$.
TO PROVE: $BC = AD$.



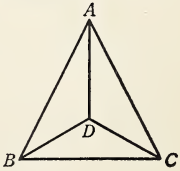
4. Are they corresponding angles in congruent triangles (s.a.s., a.s.a., or s.s.s.)?

Example. GIVEN: $AC = BC$,
$\angle BAE = \angle ABD$.
TO PROVE: $\angle AEC = \angle BDC$.



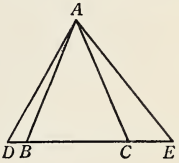
5. Are they base angles of an isosceles triangle?

Example. GIVEN: $BA = CA$,
$\angle BAD = \angle CAD$.
TO PROVE: $\angle DBC = \angle DCB$.



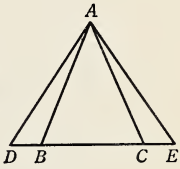
Exercises. *Using Equal Angles*

Use the questions just listed to help you prove these exercises:



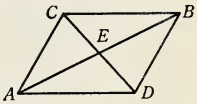
1. GIVEN: $AB = AC$.
   TO PROVE: $\angle ABD = \angle ACE$.

2. GIVEN: $AB = AC$, $BD = CE$.
   TO PROVE: $\angle D = \angle E$.

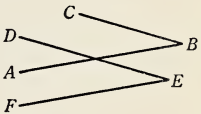


3. Give the proof that would be required for the example under question 4 on the opposite page.

4. GIVEN: Two isosceles $\triangle ABC$, DBC with a common base BC.
   TO PROVE: $\angle ABD = \angle ACD$.



5. GIVEN: $CE = ED$, $AE = EB$.
   TO PROVE: $\triangle ACE \equiv \triangle BDE$.

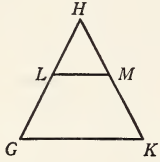


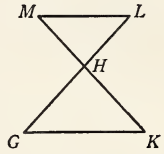
6. GIVEN: $AB \parallel FE$, $CB \parallel DE$.
   TO PROVE: $\angle B = \angle E$.

7. GIVEN: $\triangle MNP$ with $MN = MP$ and Q, R, and S the mid-points of MN, NP, and PM respectively.
   TO PROVE: $QR = SR$.

183. Profitable Questions to Ask when Trying to Prove Two Lines Equal or Find Two Lines that are Equal. Here are some more questions which you will find it helpful to study.

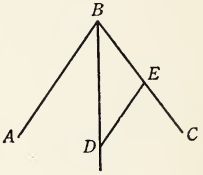
1. Are they the sums or differences of two equal pairs of lines?





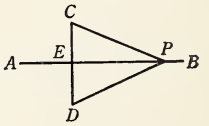
Example. GIVEN: $\triangle GHK$ with $HG = HK$, $LM \parallel GK$ and meeting HG and HK (or HG and HK produced) in L and M respectively.
TO PROVE: $LG = MK$.

2. Are they opposite two equal angles in a triangle?



Example. GIVEN: $\angle ABD = \angle CBD$, $DE \parallel AB$.
TO PROVE: $EB = ED$.

3. Are they corresponding sides in congruent triangles?

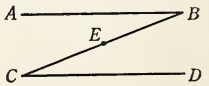


Example. GIVEN: AB bisects CD at right angles.
TO PROVE: $PC = PD$.

Exercises. *Using Equal Lines*

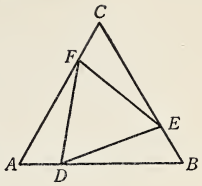
Use the questions above to help you with these exercises:

1. GIVEN: $AB \parallel CD$, $CE = EB$.



TO PROVE: E is equidistant (perpendicular distance) from AB and CD.

2. GIVEN: $AB = BC = CA$,
$AD = BE = CF$.
TO PROVE: $FD = DE = EF$.

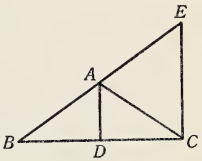


3. The lines drawn perpendicular to the sides of an isosceles triangle from the mid-point of the base are equal.

4. $ABCD$ is a parallelogram such that the diagonal AC bisects the angle C. Prove that $ABCD$ is a rhombus (a parallelogram with equal sides).

Exercises. *Miscellaneous*

1. (1) GIVEN: $\triangle ABC$ is isosceles with $AB = AC$; BA is produced to E so that $AE = AB$; EC is joined, and AD drawn $\parallel EC$.



$\angle BAD$ = corr. $\angle$ _(?)_ because _(?)_ $\parallel$ _(?)_.

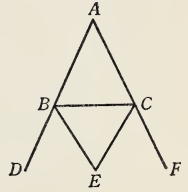
$\angle DAC$ = alt. $\angle$ _(?)_ because _(?)_ $\parallel$ _(?)_.

But _(?)_ $= BA =$ _(?)_ ; so $\triangle AEC$ is isosceles.

Then $\angle$ _(?)_ $= \angle$ _(?)_.

Thus $\angle BAD = \angle DAC$ because _(?)_.

(2) GIVEN: $AB = AC$,
BE bisects $\angle DBC$,
CE bisects $\angle FCB$.



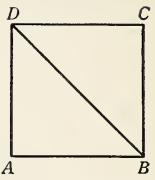
$\angle ABC = \angle$ _(?)_ because _(?)_.

$\angle DBC = \angle$ _(?)_ because _(?)_.

$\angle DBE = \angle$ _(?)_ $= \angle$ _(?)_ $= \angle$ _(?)_ because _(?)_.

$BE =$ _(?)_ because _(?)_ $=$ _(?)_.

(3) GIVEN: $ABCD$ is a square.



$AB = _(?)_ = _(?)_ = _(?)_$.

$\triangle DCB$ is $_(?)_$ because $_(?)_ = _(?)_$.

$\angle CDB = \angle _(?)_$ because DC and AB are $_(?)_$.

$\angle DCB = \angle _(?)_$ because $_(?)_$.

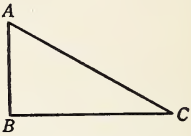
2. Which of the three endings makes the correct definition of a median?

A *median* is a line

(1) from the base to the vertex of the triangle.

(2) bisecting the sides of a triangle.

(3) from the vertex of a triangle to the middle point of the opposite side.



3. GIVEN: $\angle B = 90°$,
$\angle A =$ twice $\angle C$.
TO PROVE: $AC = 2\,AB$.

HINT. How many degrees will there be in $\angle C$? in $\angle A$? Draw a line from B to AC forming an angle of 60° with BA.

4. Given a straight line AB and a point X outside it. From X draw a perpendicular XY to meet AB in Y. Make angles YXM and YXN on opposite sides of YX and each equal to 30°. Let the arms of these angles cut AB at M and N respectively. Prove that $\triangle XMN$ is equilateral.

5. ABC is an isosceles triangle on the base BC. BA is produced to D with $AD = AB$. Prove that $\angle BCD = 90°$.

6. ABC is a right-angled triangle with a right angle at C. $\angle ACD$ is drawn equal to $\angle A$, D being the point where the arm of the angle meets AB. Prove that CD is a median and that it is equal to $\frac{1}{2}\,AB$.

7. In the right-angled $\triangle ABC$, $\angle B = 2\angle C$. Divide $\triangle ABC$ into two isosceles triangles.

CHAPTER XIV

PARALLELOGRAMS

184. Review of Special Plane Figures. Certain characteristics of quadrilaterals are laid down by definition. These characteristics are necessary and sufficient to give a figure of the type defined. There are other characteristics that these figures take on because of their defined nature. Keep clearly in mind the defined characteristics of each type of figure.

1. A *polygon* is a plane figure bounded by straight lines.

2. A *quadrilateral* is a polygon that has four sides.

3. A *parallelogram* is a quadrilateral whose opposite sides are parallel.

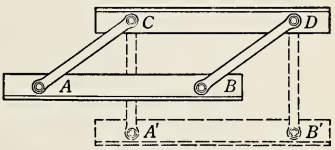
4. A *rhombus* is a parallelogram having two adjacent sides equal.

5. A *rectangle* is a parallelogram having a right angle.

6. A *square* is a rectangle having two adjacent sides equal.

7. A *trapezium** is generally defined as a quadrilateral with two sides parallel.

185. Parallel Rulers. For rapid, accurate drawing of parallel lines and parallelograms parallel rulers are very useful. In the figure at the right the two parallel sections of the ruler are held together



*Since this word is sometimes applied to a different figure, you should take care the meaning is clear in any given discussion.

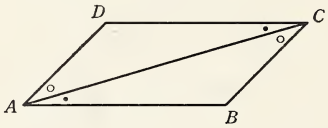
by the metal strips AC and BD. Since AC and BD are equal in length, and AB and CD are of equal length, it is always true that the edges of CD and AB are parallel. This will be proved in Theorem 18. Parallel rulers are extensively used by engineers, architects, and others who have frequent occasion to draw parallel lines.

186. Four Important Properties of Parallelograms. The following theorem will introduce you to some important facts about parallelograms:

THEOREM 17

A. *The opposite sides of a parallelogram are equal.*
B. *The opposite angles of a parallelogram are equal.*
C. *Each diagonal bisects the parallelogram.*
D. *The diagonals of a parallelogram bisect each other.*

PARTS A, B, AND C



GIVEN: $ABCD$ is a quadrilateral with $AB \parallel DC$ and $AD \parallel BC$.

REQUIRED: PART A. To prove $AB = DC$ and $AD = BC$.

PART B. To prove $\angle ADC = \angle ABC$ and $\angle DAB = \angle DCB$.

PART C. To prove that AC and DB each bisect the parallelogram $ABCD$.

PLAN: Draw one diagonal. Use Theorem 6 to obtain the facts necessary for applying Theorem 12.

CONSTRUCTION: Join AC.

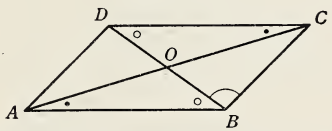
PROOF:

STATEMENTS	REASONS
In $\triangle DAC$, BCA,	
1. $\angle DCA = \angle BAC$.	1. Why?
2. $\angle DAC = \angle BCA$.	2. Why?
3. $AC = AC$.	
4. $\triangle DAC \equiv \triangle BCA$; and so $DA = BC$ and $AB = CD$.	4. Theorem 12.

This means that the opposite sides are equal and also that AC bisects the parallelogram.

5. Similarly DB bisects the parallelogram.	5. As an exercise, write out the full proof for this statement.
6. Combining statements 1 and 2, $\angle DCB = \angle DAB$.	6. Equals are added to equals, hence sums are equal.
7. Similarly $\angle ADC = \angle ABC$.	

PART D



GIVEN: Same facts as in Parts A, B, and C.

REQUIRED: To prove that DB and AC bisect each other.

PLAN: Test whether the four small triangles formed by joining the diagonals are congruent in pairs.

CONSTRUCTION: Join DB, intersecting AC in O.

PROOF:

STATEMENTS	REASONS
In $\triangle DOC$, AOB,	
1. $\angle DCO = \angle BAO$, $\angle CDO = \angle ABO$.	1. Theorem 6.
2. $DC = AB$.	2. Proved in Part A.
3. $\triangle DOC \equiv \triangle AOB$; and so $CO = OA$, and $DO = OB$.	3. Theorem 12.

187. Meaning of *Corollary*. A *corollary* is a statement that follows directly from another statement. If we accept the first statement as true, the truth of the corollary seems obvious.

COROLLARY 1 (THEOREM 17). *If one angle of a parallelogram is a right angle, all the angles must be right angles.*

As you know, such a parallelogram is called a *rectangle.*

COROLLARY 2 (THEOREM 17). *If two adjacent sides of a parallelogram are equal, all its sides are equal.*

Such a parallelogram is called a *rhombus.*

QUIZ: 1. In Theorem 17 are you given that $ABCD$ is a parallelogram or are you asked to prove it?

2. Are you given opposite sides equal or are you to prove them equal?

3. In the ordinary parallelogram are all four sides equal? In what special parallelograms is this true?

4. Could you have a parallelogram with two sides each .1 inch in length and the other two sides 50 yards in length?

5. How many acute angles are there in an ordinary parallelogram? How many in a rhombus? in a rectangle? in a square?

6. The diagonals of a parallelogram $ABCD$ intersect in O. Are OA, OB, OC, and OD all equal?

7. Statement 7 in Theorem 17 reads "Similarly $\angle ADC = \angle ABC$." (1) What change would you make in the construction in order to prove Statement 7? (2) Make the change in your notebook and prove that $\angle ADC = \angle ABC$.

8. Draw a parallelogram $PQRS$. Prove Theorem 17.

Avoid using the same letters in any one theorem so often that you are in difficulty when other letters have to be used.

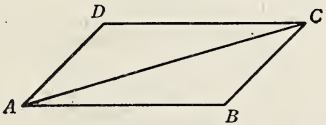
188. More Facts about Parallelograms. If a quadrilateral is constructed with opposite sides equal and no other precautions taken, will the quadrilateral necessarily be a parallelogram? Theorem 18 says *Yes*. There are three additional questions of this same form that can be answered in the same way. After Theorem 18 has been proved, refer back to this section and give the three questions.

THEOREM 18

A quadrilateral is a parallelogram

A. *If both pairs of opposite sides are equal.*
B. *If both pairs of opposite angles are equal.*
C. *If the diagonals bisect each other.*
D. *If one pair of opposite sides are equal and parallel.*

PART A



GIVEN: Quadrilateral $ABCD$ with $AB = DC$, $AD = BC$.

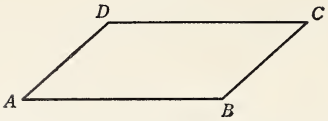
REQUIRED: To prove $AB \parallel DC$, $AD \parallel BC$.

PLAN and CONSTRUCTION: Join AC and test triangles ADC, ABC for congruency.

PROOF:

STATEMENTS	REASONS
1. In $\triangle ADC$, CBA, $AD = BC$, $DC = AB$, $AC = AC$.	1. Given.
2. $\triangle ADC \cong \triangle CBA$, and so $\angle DCA = \angle BAC$, $\angle DAC = \angle BCA$.	2. Theorem 13.
3. $DC \parallel AB$, $AD \parallel BC$.	3. Theorem 6*a*.

PART B



GIVEN: Quadrilateral $ABCD$ with $\angle A = \angle C$, $\angle D = \angle B$.

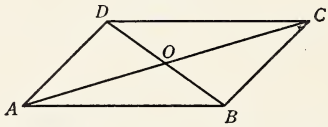
REQUIRED: To prove $AD \parallel BC$, $AB \parallel DC$.

PLAN: Find the value of two adjacent angles.

PROOF:

STATEMENTS	REASONS
1. $\angle A + \angle B + \angle C + \angle D = 360°$.	1. Why? (See Ex. 1, p. 269.)
2. $2\angle A + 2\angle D = 360°$.	2. By subst., for $\angle A = \angle C$, $\angle D = \angle B$.
3. $\angle A + \angle D = 180°$.	3. Equals ÷ equals.
4. $AB \parallel DC$.	4. Theorem 7*a*.
5. Similarly $AD \parallel BC$.	

PART C



GIVEN: Quadrilateral $ABCD$ with diagonals AC, BD intersecting at O and making $AO = OC$, $DO = OB$.

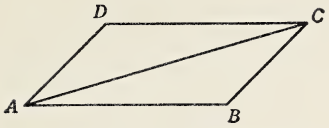
REQUIRED: To prove $AD \parallel BC$, $AB \parallel DC$.

PLAN: Test congruency of $\triangle DOC$, BOA.

PROOF:

STATEMENTS	REASONS
1. In $\triangle DOC$, BOA, $AO = OC$, $DO = OB$.	1. Given.
2. $\angle DOC = \angle BOA$.	2. Theorem 5.
3. $\triangle DOC \cong \triangle BOA$, and so $\angle DCO = \angle BAO$.	3. Theorem 11.
4. $AB \parallel DC$.	4. Theorem 6*a*.
5. Similarly $AD \parallel BC$.	

PART D



GIVEN: Quadrilateral $ABCD$ with $AB = DC$ and $AB \parallel DC$.

REQUIRED: To prove $AD \parallel BC$.

PLAN and CONSTRUCTION: Join AC and test congruency of $\triangle CAD$ and CAB.

PROOF:

STATEMENTS	REASONS
In $\triangle CAD$ and CAB,	
1. $\angle DCA = \angle BAC$.	1. Theorem 6.
2. $CD = AB$.	2. Given.
3. $AC = AC$.	
4. $\triangle CAD \cong \triangle CAB$.	4. Theorem 11.
5. $\angle CAD = \angle ACB$.	
6. $AD \parallel BC$.	6. Theorem 6*a*.

QUIZ: 1. In Theorem 18, are you given that $ABCD$ is a parallelogram?

2. Are you given that opposite sides are equal or are you to prove that they are equal?

3. Give the hypothesis in the enunciation of Theorem 18.

4. Draw a quadrilateral in which only one pair of opposite sides are equal.

5. Draw a quadrilateral in which only one pair of opposite angles are equal.

6. In a triangle equality of the three sides means necessarily that the three angles are equal. If the angles are equal, are the sides necessarily equal?

7. In a quadrilateral equality of sides does not mean necessarily that the angles are equal.

Which quadrilateral has equal sides and equal angles? Which quadrilateral has equal sides but not all angles equal? Which quadrilateral has equal angles but not all sides equal? Which quadrilateral has two pairs of equal sides and two pairs of equal angles?

189. Language Practice. A problem may be expressed either with or without reference to a particular lettered figure. For example, the two following problems are identical:

1. (With reference to a lettered figure.) ABC is a triangle in which the altitude BE equals the altitude CF. Prove that $AB = AC$.

2. (Without reference to a lettered figure.) If the altitudes drawn to two sides of a triangle are equal, the triangle is isosceles.

Exercises. *Language Practice*

Express each of the following problems in correct English without reference to a lettered figure. No proofs are required.

1. In any triangle ABC, $AB + BC$ is greater than AC.

2. In any triangle ABC, any side AB is less than $\frac{1}{2}(AB + BC + CA)$.

3. $ABCD$ is a parallelogram. The four bisectors of the angles A, B, C, and D form a rectangle.

4. $ABCD$ is a parallelogram. Equilateral triangles ABE, BCF, DCG, and ADH are described outside the parallelogram. Prove that $EFGH$ is a parallelogram.

5. The bisectors of the interior angle at C and the exterior angles at A and B of $\triangle ABC$ meet at a point.

6. Construct a triangle ABC having given the size of $\angle B$, C, and the length of the perpendicular from A to BC.

7. E, F, G, and H are the mid-points of the sides AB, BC, CD, and DA of the rhombus $ABCD$. Prove that $EFGH$ is a rectangle.

8. $ABCD$ is a parallelogram. Prove that the perpendiculars from A and C to BD are equal.

State each of the following problems with reference to a particular lettered figure. Sketch the figure. No proofs are required.

9. Prove that the perimeter of a quadrilateral is greater than the sum of the diagonals.

10. On a given straight line as base construct an isosceles triangle.

11. If the diagonals of a quadrilateral divide it into four equal triangles, the quadrilateral is a parallelogram.

12. Find a point equidistant from two given points and at a fixed distance from a given straight line.

13. The sum of the squares on the diagonals of a rhombus is four times the square on any one of its sides.

14. Divide a parallelogram into two equal areas by a straight line through a given point.

15. A quadrilateral has two angles which are right angles. Prove that the other two angles are supplementary.

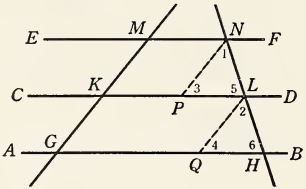
16. The diagonals of a rhombus bisect each other at right angles.

17. The lines drawn from the mid-points of the equal sides of an isosceles triangle to the mid-point of the third side of the triangle are equal.

190. Transversals and Parallels. When transversals cut parallel straight lines, there are many interesting relations among the different line-segments formed.

THEOREM 19

If three or more parallels intercept equal segments on one transversal, they intercept equal segments on every other transversal.



GIVEN: The parallel lines AB, CD, EF cut by two transversals at G, K, M and H, L, N respectively, and $NL = LH$.

REQUIRED: To prove $MK = KG$.

PLAN and CONSTRUCTION: Draw NP and LQ parallel to MG, as in the figure. Test congruency of $\triangle NPL$, LQH.

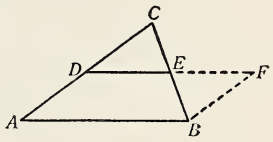
PROOF:

STATEMENTS	REASONS
1. NP is parallel to LQ.	1. NP, LQ each $\parallel MG$.
In $\triangle NPL$, LQH,	
2. $NL = LH$.	2. Given.
3. $\angle 1 = \angle 2$.	3. Theorem 8.
4. $\angle 5 = \angle 6$.	4. Theorem 8.
5. $\therefore \triangle NPL \equiv \triangle LQH$, and $NP = LQ$.	5. Theorem 12.
6. $NP = MK$, $LQ = KG$.	6. Theorem 17.
7. $MK = KG$.	7. They are equal to equal quantities.

COROLLARY (THEOREM 19). *The straight line drawn through the middle point of one side of a triangle parallel to another side bisects the third side.*

Exercises. *Parallels*

1. *An important exercise.* The straight line joining the middle points of two sides of a triangle is parallel to the third side and equal to half of it.



HINT. Draw $BF \parallel AC$. Produce DE to F.

Compare triangles DEC, FEB, using Theorem 12.

Prove that $ABFD$ is a parallelogram. Use the fact that EF has been proved equal to DE.

2. The line which joins the mid-points of two adjacent sides of any quadrilateral is equal and parallel to the line which joins the mid-points of the other two sides.

3. The lines which join the mid-points of the sides of a triangle divide it into four congruent triangles.

4. The diagonals of a rhombus bisect the angles through which they pass.

191. Summary. In this chapter you have learned

(1) What the distinguishing features of various types of quadrilaterals are.

(2) Theorems:

THEOREM 17.

A. *The opposite sides of a parallelogram are equal.*

B. *The opposite angles of a parallelogram are equal.*

C. *Each diagonal bisects the parallelogram.*

D. *The diagonals of a parallelogram bisect each other.*

COROLLARY 1. *If one angle of a parallelogram is a right angle, all the angles must be right angles.*

COROLLARY 2. *If two adjacent sides of a parallelogram are equal, all its sides are equal.*

THEOREM 18. *A quadrilateral is a parallelogram*

A. *If both pairs of opposite sides are equal.*

B. *If both pairs of opposite angles are equal.*

C. *If the diagonals bisect each other.*

D. *If one pair of opposite sides are equal and parallel.*

THEOREM 19. *If three or more parallels intercept equal segments on one transversal, they intercept equal segments on every other transversal.*

COROLLARY. *The straight line drawn through the middle point of one side of a triangle parallel to another side bisects the third side.*

EXERCISE. *The straight line joining the middle points of two sides of a triangle is parallel to the third side and equal to half of it.*

Test on Chapter XIV

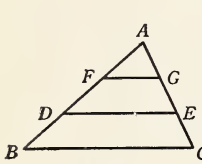


FIG. 1

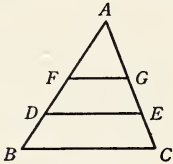


FIG. 2

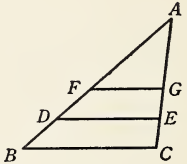


FIG. 3

In these three figures $BC \parallel DE \parallel FG$

1. In Fig. 1, given that $AB = 6$ inches and that $AG = GE = EC$, what conclusions may be drawn?

2. In Fig. 2, given that $AB = 10$ inches and that $GE = EC = 2\frac{1}{2}$ inches, list the other facts that are known.

3. In Fig. 3, given that $GE = EC = 2$ inches and that $AF = 7$ inches, list the other facts that are known.

4. State the meaning of the following words:

rhombus trapezium parallelogram corollary transversal

5. Think of the difference between the following pairs of figures and name the fewest changes that must be made in order

(1) to change a parallelogram with unequal adjacent sides and angles into a rhombus; a square; a rectangle,

(2) to change a square into a rhombus,

(3) to change a rhombus into a rectangle.

6. Prove that if the mid-points of the adjacent sides of a quadrilateral are joined, the figure thus formed is a parallelogram.

7. Prove that the straight lines joining the mid-points of opposite sides of a quadrilateral bisect each other.

8. Copy the table and write the answers (*Yes* or *No*) for each of the ten exercises. You will write forty answers.

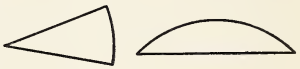
		In All Parallelograms	In All Squares	In All Rectangles	In All Rhombuses
(1)	The opposite sides are equal.	(?)	(?)	(?)	(?)
(2)	The diagonals are equal.	(?)	(?)	(?)	(?)
(3)	The opposite angles are equal.	(?)	(?)	(?)	(?)
(4)	All the angles are equal.	(?)	(?)	(?)	(?)
(5)	The diagonals bisect each other.	(?)	(?)	(?)	(?)
(6)	All the sides are equal.	(?)	(?)	(?)	(?)
(7)	The diagonals intersect at right angles.	(?)	(?)	(?)	(?)
(8)	All angles are right angles.	(?)	(?)	(?)	(?)
(9)	The opposite sides are parallel.	(?)	(?)	(?)	(?)
(10)	The diagonals bisect the angles.	(?)	(?)	(?)	(?)

CHAPTER XV

SIMILARITY AND SYMMETRY

1. Similarity

192. Straight-Line and Curved-Line Figures. Most of the figures which you have drawn in your work in geometry have been straight-line figures such as triangles, quadrilaterals, and other polygons.



In your early work you used compasses to draw arcs and circles. You have probably drawn figures such as the two above in which both straight and curved lines appear.

193. Regular and Irregular Figures. A regular polygon has all its sides and angles equal. A regular triangle is thus equilateral, and a regular quadrilateral is a square. Study the diagrams below so that you may understand the difference between regular and irregular figures.

Regular figures

Irregular figures

194. Shape of Figures. In Theorems 11, 12, and 13 the triangles were in each instance of the same shape; but if you will turn to Theorem 21, page 334, you will see in the diagram two triangles whose shapes vary greatly. When both size and shape of rectilinear figures are identical, as in Theorem 11, the figures are said to be *congruent*, as you already know.

When two or more figures have the same shape but not the same size, they are said to be *similar*. Such figures differ only in the scale used in drawing them. The following pairs of figures are similar.

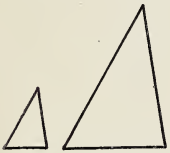


FIG. 1

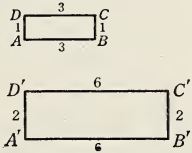


FIG. 2

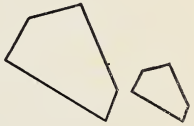


FIG. 3

When two figures are similar, (1) their corresponding angles are equal, and (2) the ratio of any two corresponding sides equals the ratio of any other pair of corresponding sides. In Fig. 2 above, the corresponding angles are evidently equal, and $\frac{AB}{A'B'} = \frac{3}{6} = \frac{1}{2}$, $\frac{BC}{B'C'} = \frac{1}{2}$, or $\frac{AB}{A'B'} = \frac{BC}{B'C'}$.

When two figures are congruent, we may say that the scale used is 1 : 1.

195. Constructing Similar Figures. We have already mentioned that similar figures may be produced by changing the scale of the drawing. There are other methods of producing similar figures. Either the *double compasses* or the *pantograph* may be used for reproductions.

196. Double Compasses. *AB*, *CD* in Fig. 1 form a pair of compasses. The two legs are clamped at *E*. By making $DE = \frac{1}{2}EC$ and $BE = \frac{1}{2}EA$, line-segments in one figure may be transferred to twice or to one half their original length. In Fig. 2 two triangles are shown. These illustrate the changes made when the 2 : 1 ratio holds.

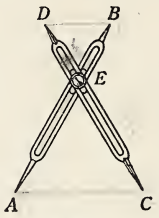


FIG. 1

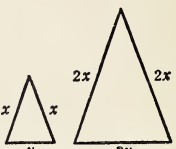


FIG. 2

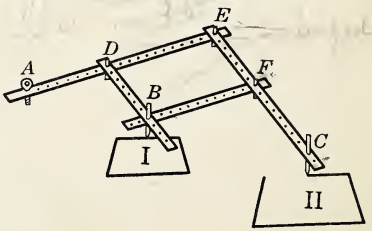
If *EC* is made three times *ED* and *AE* is made three times *EB*, the figure when reproduced will have its sides either three times or one third as long as in the original figure.

Note that if *DB* and *AC* are joined in Fig. 1, there are two similar figures formed, namely $\triangle DEB$ and $\triangle AEC$.

How must the clamp be set to reproduce a figure which will have its linear parts $2\frac{1}{2}$ times their original length?

197. The Pantograph. The pantograph is an instrument used for enlarging or reducing maps, drawings, and designs.

The four bars *AE*, *EC*, *BF*, and *BD* form a parallelogram *BFED*. The bars are hinged at *E*, *F*, *B*, and *D*. The point *A* is fixed securely and does not move while the instrument is in use. *B*, to which is attached a small pointed tracer, moves over the outline to be reproduced. The point *C*, which is a pencil, traces a figure similar to the one which *B* is tracing.



The instrument may be set to give a reproduction on any desired scale. The linear scale of the sides of the new figure is always (length of AE) ÷ (length of AD).

As C moves, the shape of the parallelogram $BDEF$ changes. This shows that a parallelogram is not fixed when its four sides are given.

Exercises. *Similar Figures*

1. Draw a line-segment 6 inches long. Then draw it to scale, using in succession the scales 1:2; 1:3; 1:4; 1:8; 1:10; 1:12.

2. In a scale drawing a certain line-segment was found to be 2 inches long. What was the actual length represented by this segment if the scale used was 1:2? 1:4? 1:10? 1:12? 1:100?

3. Suppose that two figures, such as those shown in the diagram, are similar. The table below gives the sides of Fig. 1 and also the side of Fig. 2 which corresponds to AB. Find the missing values.

	Length of Sides				
Fig. 1	2	3	$3\frac{1}{2}$	5	4
Fig. 2	4				

Hint. $\frac{4}{2}=\frac{?}{3}$; $\frac{4}{2}=\frac{?}{3\frac{1}{2}}$; etc.

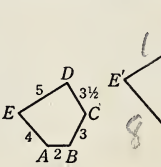


Fig. 1

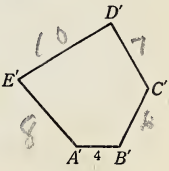
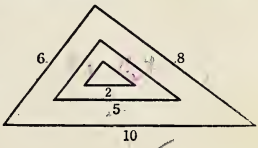


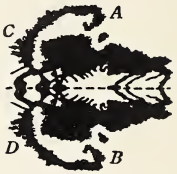
Fig. 2

4. The three triangles shown in the figure are similar. In each of the smaller triangles the length of the side corresponding to 10 is given. Find the lengths of the other two sides in each of the smaller triangles.



2. Symmetry

198. Symmetrical Figures. Drops of ink were placed near a crease that had been made in a sheet of paper. The paper was re-folded along the same crease while the ink was wet. The ink-blot figure here shown was the result.



Try writing your name exactly on the crease line and reproducing this by the *blot* method.

Such figures illustrate a principle known as "symmetry." Top and bottom halves, or right and left halves, *correspond*. For each point in one half there is an image corresponding to it in the other half.

The real or imaginary line dividing the two halves is called the *axis of symmetry*.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

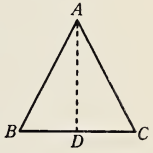
Fig. 6

There are three kinds of symmetry, namely: (1) *point* symmetry, (2) *line* symmetry, and (3) *plane* symmetry.

1. *Point Symmetry.* A circle is a good example of *point* symmetry. The centre of a circle is the *mid-point* of every diameter of the circle. It is called the *centre of symmetry.* Figs. 2 and 5 on page 316 illustrate point symmetry.

This kind of symmetry is also known as *central* symmetry.

2. *Line Symmetry.* An isosceles triangle illustrates line symmetry. Every isosceles triangle, such as ABC in the figure, may be folded on a line AD so that the two parts into which the triangle is divided by that line can be made to fall together, or coincide. The line of folding is also known as the line of symmetry, or the axis of symmetry. Notice how Figs. 1, 2, 3, and 5 illustrate this type of symmetry.



This kind of symmetry is often called *axial* symmetry.

3. *Plane Symmetry.* If you cut an orange in half, the flat surface of either half lies in a plane which is called the plane of symmetry. If the orange were a perfect sphere, then each half would be an exact counterpart of the other half. The electric-light bulb and jar shown in Figs. 4 and 6 on the opposite page are good illustrations of this type of symmetry.

It will be seen that in every case of symmetry the whole figure shows a perfect balance or harmony with reference to a point, or to a line, or to a plane.

Exercises. *Everyday Uses of Symmetry*

1. A modern home suggests many illustrations of symmetry. These may be found in the methods of construction, in the placing of doors and windows, of furniture, of rugs and pictures, in the forms of chairs, desks, and kitchen utensils, in the patterns or designs used in draperies, curtains, woodwork, and the like. Find at least ten illustrations of symmetry (1) in your home; (2) in the schoolroom.

2. The figure at the right represents one half of the front view of a house. Copy the figure and sketch the other half of the house.

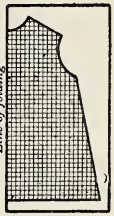
3. Visit a modern department store, or an office building, and try to discover applications of symmetry in its plan of construction and its equipment.

4. Look through a copy of any magazine containing advertisements. How many examples of each kind of symmetry can you find?

5. In the picture of a vase shown at the right, which kind of symmetry is suggested? How many planes of symmetry does this vase have?

6. The human body has a plane of symmetry. Think of the position of the eyes or the ears, and of the hands and feet when you stand erect. This fact is of importance in the making of clothing, shoes, and gloves. Explain.

7. Betty watched her mother use a paper pattern in making a dress for the baby. Then she tried to use the same method in making a dress for her doll. Can you explain the use of symmetry in this method?



8. Explain the practical use of line symmetry in tailoring.

9. The capital letter T illustrates what kind of symmetry? Explain.

10. Sketch each of the following capital letters: A, E, H, M, V, U, W, X, Y. Show that each is an illustration of line symmetry.

11. Observe that a rectangle has two lines of symmetry. Fold a rectangular sheet of paper so as to show each of these lines.

12. Fold a sheet of paper. Then fold again so as to place one part of the crease on the other part. Observe that the second line of folding is *perpendicular* to the first. (Why?) There are now four layers of paper. (Why?) Now cut across the folded paper, from one crease to the other. Open the paper. The resulting figure is seen to have *two* lines of symmetry. Explain.

13. By using a method similar to that of Exercise 12, try to make "paper lace" having two, three, or more lines of symmetry.

14. Fold a sheet of paper. Cut out any part of the paper, beginning at the crease and returning to it. Open the paper. What kind of symmetry do you observe in the resulting figure?

15. Point out examples of symmetry in your schoolroom.

199. Two Important Facts regarding Symmetry. Of the several facts we have noted regarding symmetrical figures, the following are very important:

1. If for a given figure there is a centre of symmetry, then for every point P in the figure there is a corresponding point Q such that PQ is bisected at the centre of symmetry.

2. If for a given figure there is an axis of symmetry, then for every point P in the figure there is a corresponding point Q such that PQ is perpendicular to the axis and is bisected by it.

200. Right Bisector. If one straight line bisects another straight line at right angles, the first line is called the right bisector of the other.

QUIZ: 1. What is the axis of symmetry for an isosceles triangle?

2. What are the axes of symmetry for an equilateral triangle?

3. What type of quadrilateral has symmetry about a line?

4. Does any type of quadrilateral have symmetry about a point?

5. For any given circle is there a centre of symmetry? is there an axis of symmetry?

6. How many axes of symmetry has (1) a rectangle? (2) a square? (3) a regular pentagon?

201. Summary. In this chapter you have learned the meaning of each of the following:

(1) regular and irregular figures
(2) similar figures
(3) symmetry
(4) point symmetry
(5) line symmetry
(6) plane symmetry
(7) corresponding points in symmetrical figures
(8) right bisector

Test on Chapter XV

1. Which geometric idea is illustrated by (1) a photograph and its enlargement? (2) two golf balls? (3) two one-cent stamps? (4) two sheets of paper from the same pad?

2. The ratio of 9 to 5 is the same as the ratio of _(?)_ to 100.

3. On a scale drawing 1 inch is to represent 24 feet. How long a line will be necessary to picture 54 feet?

4. Two quadrilaterals are similar. The sides of the first are 4, $4\frac{1}{2}$, 5, and 6. In the second quadrilateral the side corresponding to 4 is 2. Find the remaining sides.

5. A daisy illustrates _(?)_ symmetry.

6. A butterfly illustrates _(?)_ symmetry.

7. An object reflected in a mirror illustrates _(?)_ symmetry.

8. Sketch each of the following figures, and in each figure draw all its lines of symmetry.

CHAPTER XVI

CONSTRUCTIONS; AIDS IN EXERCISE-SOLVING

202. Planning Solutions. Asking yourself questions such as those outlined below will help you to decide what plan to use in solving an exercise.

1. To prove line-segments equal: (Refer to section 183.)
   (1) Are they opposite sides of a parallelogram?
   (2) Are they the lines drawn from a point in a perpendicular bisector of a line to the ends of the line?
   (3) Are they the segments cut off on a transversal by parallel lines that are known to cut off equal segments on another transversal?
2. To prove angles equal: (Refer to section 182.)
   (1) Are they opposite angles of a parallelogram?
   (2) Are they corresponding angles of similar figures?
3. To prove lines parallel:
   (1) Are they perpendicular to the same line?
   (2) Are they parallel to the same line?
   (3) Are they opposite sides of a parallelogram?
   (4) Does one line join the mid-points of two sides of a triangle of which the other line is the third side?
4. To prove that a quadrilateral is a parallelogram:
   (1) Are both pairs of opposite sides parallel?
   (2) Are both pairs of opposite angles equal?
   (3) Is one pair of opposite sides equal and parallel?
   (4) Do the diagonals bisect each other?

203. An Additional List of Aids in Solving Exercises in Geometry. Look back to section 165. Re-read the list of helps given there. To that list we now add the following:

1. Exercises often begin with some part of a figure given. For example, (1) *Divide any given angle into* · · ·, (2) *Divide a given straight line* · · ·, (3) *ABC is a given isosceles triangle with the vertex joined to the middle point of the base. Prove* · · ·, or (4) *The diagonals of a given quadrilateral bisect each other at right angles. Prove* · · ·.

The first step towards a solution is the drawing of the given parts of the figure. For the above they would be:

FIG. 1

FIG. 2

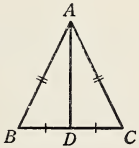


FIG. 3

FIG. 4

2. It is an advantage to mark parts that are given equal. In Fig. 3 the single check marks on BD and DC indicate that they are equal. The double check marks on AB and AC indicate that they are equal. As you proceed with your proof, mark any additional parts found to be equal.

3. As soon as you know what you are to prove, stop long enough to think of the theorems that are likely to be of assistance. If, for example, you are to prove two lines parallel, you think: *I can prove the lines parallel if I know that*

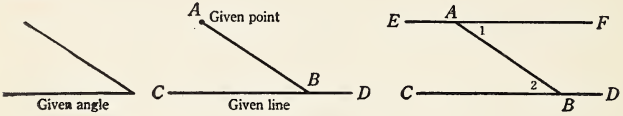
(1) *Two alternate interior angles are equal,* or

(2) *Two corresponding angles are equal,* or

(3) *Two interior angles on the same side of the transversal are together equal to two right angles,* and so on.

Having recalled the related theorems, study the figure carefully to find whether any of these conditions apply.

4. Draw freehand the finished construction, assuming that what is to be proved is true or that what was to be constructed has been constructed. Study the figure and list the other truths that are necessarily true in the light of this fact. For example, consider this problem: *Find a point B in a given straight line CD such that if AB be drawn to B from a given point A, the angle ABC will be equal to a given angle.*



Begin by assuming that the point B has been found as in the second figure.

The third figure suggests that a line through A parallel to CD might prove useful, and we notice that $\angle 1 = \angle 2$.

If we have a line drawn through A parallel to CD, we can construct $\angle 1$ equal to the given angle. One arm of $\angle 1$ will cut the line CD at the point B, forming $\angle 2 = \angle 1$.

We can draw a line through A parallel to CD, and so we see the solution of our problem.

The rough drawing is now discarded, and the solution proceeds as follows:

(1) Draw the given parts of the figure.

(2) Name the point A and the line CD.

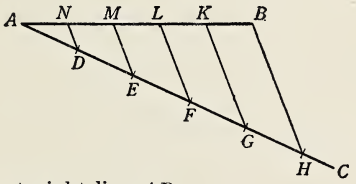
(3) Draw a line through A parallel to CD. Mark the line as EAF.

(4) Construct $\angle FAB$ equal to the given angle, letting the arm of the angle cut CD at B. *B is the required point.*

204. Dividing a Line into Equal Parts. The facts established in Theorem 19 may be used to divide a given straight line into any number of equal parts.

CONSTRUCTION 7

To divide a given straight line-segment into any number of equal parts.



GIVEN: A straight line AB.

REQUIRED: To divide AB into any number of equal parts.

PLAN: Take five as the number of equal parts. Construct a figure to which Theorem 19 is applicable.

CONSTRUCTION: 1. At any convenient angle with AB draw from A a line-segment AC.

2. Choose a convenient radius and lay off on AC five successive segments AD, DE, EF, FG, and GH each equal to this radius.

3. Join HB.

4. From G, F, E, and D draw parallels to HB intersecting AB in K, L, M, and N.

PROOF:

STATEMENTS	REASONS
1. $AD = DE = EF = FG = GH$.	1. Construction.
2. $AN = NM = ML = LK = KB$.	2. Theorem 19.

QUIZ: 1. Why is it not possible after choosing the radius to lay off these segments on AB instead of on AC?

2. Do you see any relation between Construction 7 and the corollary of Theorem 19?

3. If line AB, which is 10 inches in length, is divided into n equal parts, how long is each of these parts?

4. In the figure for Construction 7,

(1) Is AD of fixed or of variable length?

(2) Must AD be less than, equal to, or greater than AN?

(3) Should the line AC fall above or below AB?

(4) What size should $\angle BAC$ be?

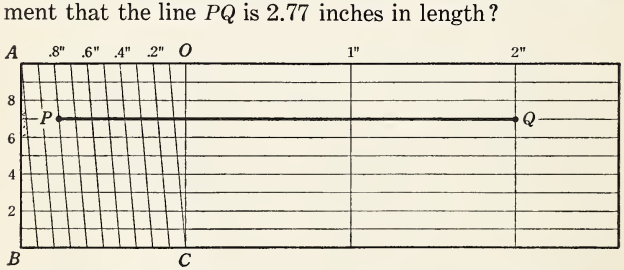
(5) Might the angle which is added to the given figure be constructed at B as $\angle ABC$ instead of at A as $\angle BAC$?

5. Have these two expressions the same meanings?

(1) Cut off three equal segments from a line.

(2) Divide a line into three equal segments.

205. Diagonal Scale. The *diagonal scale* may be used to measure lengths to hundredths of an inch. OA, AB, and BC are each divided into tenths. Can you verify the statement that the line PQ is 2.77 inches in length?



206. Studying Several Special Cases of One Exercise. It is interesting to study various figures which may be drawn in connection with an exercise. In this section we shall

study several cases that may arise in this problem: *A and B are any two points on the same side of the straight line CD; find a point P in CD such that when PA, PB are drawn, these lines will make equal angles with CD.*

A and *B* are any two points *on the same side* of the line *CD*. It is required to find a point *P* in the line *CD* such that $\angle APC = \angle BPD$.

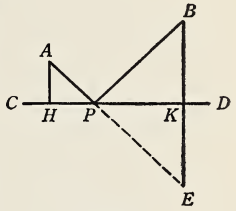


FIG. 1

In Figs. 1, 2, and 3, the points *A*, *B*, which may be any two points, have been taken in different relative positions.

In Fig. 1, *A* is nearer *CD* than is *B*.

In Fig. 2, *A* and *B* are equidistant from *CD*.

In Fig. 3, *B* is nearer *CD* than is *A*.

In each case the following construction is completed: *BK* is drawn $\perp$ *CD*; *BK* is produced to *E* with $KE = BK$; *AE* is joined.

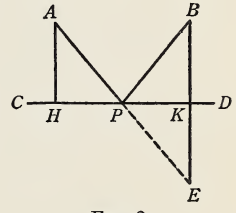


FIG. 2

Proof: $\triangle BPK \equiv \triangle EPK$ (Theorem 11),

$$\angle BPK = \angle KPE = \angle APC.$$

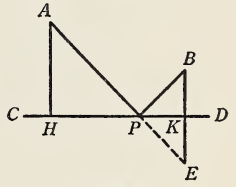


FIG. 3

QUIZ: 1. If *H* and *K* are the feet of the perpendiculars from *A* and *B*, what may be said about the position of *P* regardless of the distances of *A* and *B* from *CD*?

2. Where is *P* when *A* and *B* are equidistant from *CD*?

3. Where is *P* when *A* is nearer *CD* than is *B*?

4. Where is *P* when *B* is nearer *CD* than is *A*?

5. Where is P when A is about 1 inch from CD and B is about 100 feet from CD?

6. Where is P when A is about 100 feet from CD and B is about 1 inch from CD?

If A remains at a fairly long distance from CD while B is very near CD, the figure takes the form of Fig. 4.

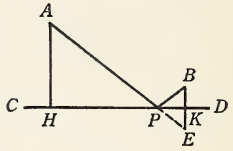


FIG. 4

If B is on the side of CD opposite to A, then the problem becomes: *A and B are any two points on opposite sides of CD; find a point P in CD such that AP and BP make equal angles with CD.* (See Fig. 5, where, when the instructions on page 327 are carried out in the construction, AE does not cut CD at P but must be produced to meet CD in P.)

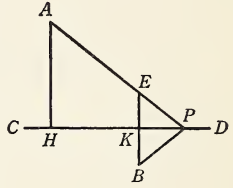


FIG. 5

QUIZ: Answer for Fig. 5 the first four questions of the quiz on page 327.

Exercises. *Construction Practice*

In the following exercises draw and letter the figures, and state what is given and what is required. No proofs are necessary.

1. A straight line drawn at right angles to BC, the base of an isosceles triangle ABC, cuts the side AB at D and CA produced at E. Show that AED is an isosceles triangle.

2. From the ends of the base of an isosceles triangle, perpendiculars are drawn to the sides. Prove that they make with the base an angle equal to half the vertex angle.

3. $ABCD$ is a parallelogram. AB and CD are produced in opposite directions to K and R so that $BK = DR$. Prove that $\triangle ADR$ and CBK are congruent.

4. The perpendicular bisectors of the sides of a triangle intersect in a common point.

5. Prove that if D is any point on AB of $\triangle ABC$, E the middle point of AD, F the middle point of AC, and G the middle point of CD, then $DEFG$ is a parallelogram.

6. ABC is a triangle with BA and CA produced to P and Q, making $AP = AB$ and $AQ = AC$. If R is any point in BC, prove that the sum of $\triangle RAP$, RAQ is constant.

7. Given $AB \parallel CD$. A point E is taken in AB produced. The sum of $\triangle CED$ and $BED = \triangle AEC$. Prove $AC = BD$.

207. Summary. In this chapter you have learned

(1) To attack problems in orderly fashion.

(2) To divide a line into equal parts.

(3) To appreciate that several special cases may arise in the solution of an exercise.

Test on Chapter XVI

1. State points to bear in mind when trying to prove (1) line-segments equal; (2) angles equal; (3) lines parallel; (4) a quadrilateral to be a parallelogram.

2. Draw a line 5 inches long and divide it into 7 equal parts.

3. Draw figures showing several cases of the following: $PQRS$ is a quadrilateral with $PQ \parallel SR$, $\angle S = \angle R$, and X, Y points in SR such that $SX = RY$. Prove $PY = QX$.

4. Construct the figure described in the following exercise. No proof is required.

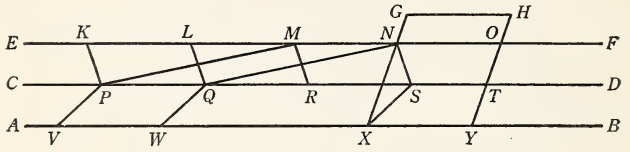
$PQRS$ is a parallelogram. PS is bisected at E and RQ at F. Prove that QS is trisected when ER and PF are drawn.

CHAPTER XVII

AREAS

208. Language Practice. The following exercises will test your understanding of the terms to be used in the next theorems.

Exercises. *Language Practice*



In this figure, $KL = LM = MN = NO = GH = PQ = QR = RS = VW = XY$; AB, CD, EF, and GH are parallel; EF and AB are equidistant from CD.

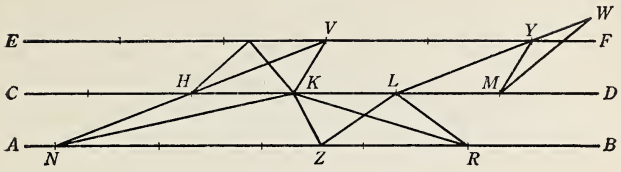
1. Name two parallelograms on the same base and between the same parallels.

2. Name two parallelograms on the same base and not between the same parallels.

3. Name five parallelograms having equal bases and the same altitude.

4. Name two parallelograms having the same base but unequal altitudes.

5. Name two parallelograms on equal bases (not the same base) and having unequal altitudes.



In this figure, AB and EF are both parallel to CD and equidistant from it; $HK = KL = LM$.

6. Name two triangles on the same base and between the same parallels.

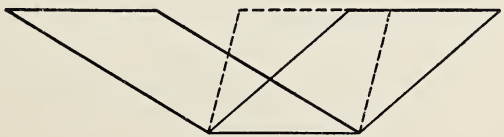
7. Name two triangles on equal bases (not the same base) and between the same parallels.

8. Name two triangles on the same base and not between the same parallels.

9. Name five triangles that have the same or equal bases and equal altitudes.

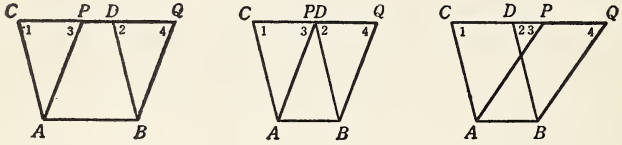
10. Name two pairs of triangles, each pair of which is on the same base with the members of the pair having unequal altitudes.

209. Different Parallelograms on the Same Base. The parallelograms in the figure below are all on the same base, but they have very different shapes. The question answered by Theorem 20 is *Have all these parallelograms the same area?*



THEOREM 20

Parallelograms on the same base and between the same parallels (or of the same altitude) are equal in area.



GIVEN: $ABDC$ and $ABQP$ are parallelograms on the same base AB and between the same parallels AB and CQ.

REQUIRED: To prove that $\|^m ABDC = \|^m ABQP$.*

PLAN: Compare $\triangle APC$ and BQD. If these can be proved equal, the differences when each is taken from the quadrilateral $BACQ$ are equal.

PROOF:

STATEMENTS	REASONS
In $\triangle APC$, BQD,	
1. $AP = BQ$.	1. Theorem 17, A.
2. $\angle 1 = \angle 2$, $\angle 3 = \angle 4$.	2. Theorem 8.
3. $\triangle APC = \triangle BQD$.	3. Theorem 12.
4. Subtract $\triangle APC$ from the figure $ABQC$. The difference is $\|^m ABQP$.	
5. Subtract $\triangle BQD$ from the figure $ABQC$. The difference is $\|^m ABDC$.	
6. $\|^m ABQP = \|^m ABDC$.	6. Equals taken from equals.

*$\|^m$ is used for *parallelogram*. Distinguish between "=" used here to mean "equal in area" and "$\equiv$" used to mean "congruent to."

COROLLARY 1 (THEOREM 20). *The area of a parallelogram is equal to the product of its base by its altitude.*

The area of the parallelogram is the same as that of a rectangle on the same base and between the same parallels.

COROLLARY 2 (THEOREM 20). *Parallelograms on equal bases and of the same altitude are equal in area.*

Exercises. *Parallelograms*

1. Express Corollary 1 as a formula. Does it make any difference which side of the parallelogram you take as the base?

2. A parallelogram has an area of 20 square inches. Its altitude is 4 inches. Draw the parallelogram. Do the given data determine

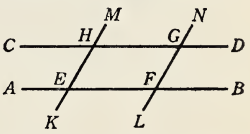
(1) the base of the parallelogram?

(2) the distance between opposite sides?

(3) the perimeter?

(4) the shape of the parallelogram?

3. (1) Between what parallels does the parallelogram *EFGH* lie?



(2) If *FG* be taken as the base of the parallelogram, draw and measure the altitude.

(3) Name a line of length different from that of *FG* which might be taken as the base. What is the altitude of the parallelogram on this base?

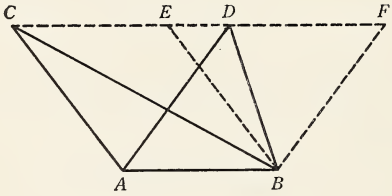
(4) Use the conditions of (2) to find the area of the parallelogram. Check by using the conditions of (3).

4. Prove that the lines joining the mid-points of the opposite sides of a parallelogram divide it into four equal parallelograms.

210. Triangles on the Same Base and with the Same Altitude. Under these conditions triangles may have widely differing shapes. What can we say about their areas?

THEOREM 21

Triangles on the same base and between the same parallels are equal in area.



GIVEN: $\triangle ABC$ and ABD on same base AB and between the same parallels AB and CF.

REQUIRED: To prove $\triangle ABC = \triangle ABD$.

PLAN and CONSTRUCTION: Complete parallelograms using BC and BD as diagonals.

PROOF:

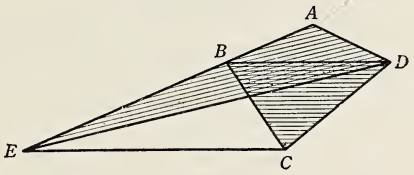
STATEMENTS	REASONS
1. $\triangle ABC = \frac{1}{2} \parallel^m ABEC$.	1. Theorem 17, C.
2. $\triangle ABD = \frac{1}{2} \parallel^m ABFD$.	2. Theorem 17, C.
3. $\parallel^m ABEC = \parallel^m ABFD$.	3. Theorem 20.
4. $\triangle ABC = \triangle ABD$.	4. They are halves of equal magnitudes.

COROLLARY (THEOREM 21). *Triangles on equal bases and of the same altitude are equal in area.*

211. Triangle Equal in Area to a Quadrilateral. It is often desirable to find a triangle that has the same area as a given rectilinear figure. Construction 8 shows how to construct a triangle equal in area to a quadrilateral.

CONSTRUCTION 8

To construct a triangle equal in area to a given quadrilateral.



GIVEN: Quadrilateral $ABCD$.

REQUIRED: To construct a triangle equal in area to $ABCD$.

PLAN: Divide the quadrilateral into two triangles. Use the fact that triangles on the same base and between the same parallels are equal. Pick the particular equivalent triangle that has one of its sides in AB produced.

CONSTRUCTION: Join BD. Through C draw a line parallel to BD. Produce AB to meet this line in E. Join ED. $\triangle AED$ is the required triangle.

PROOF:

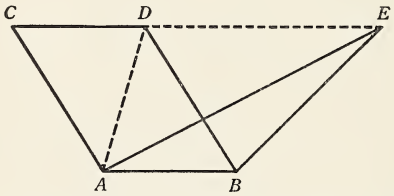
STATEMENTS	REASONS
1. Quadrilateral $ABCD = \triangle ABD + \triangle BDC$.	1. Construction.
2. $\triangle BDC = \triangle BED$.	2. Theorem 21.
3. Quadrilateral $ABCD = \triangle AED$.	3. Substituting in Statement 1.

Give in detail the construction and proof for Construction 8, retaining AB and BC in the final triangle.

212. A Relation between Triangles and Parallelograms on the Same Base. Triangles and parallelograms may stand on the same base and be between the same parallels. Theorem 22 establishes the relation that exists between these figures when so placed.

THEOREM 22

If a triangle and a parallelogram stand on the same base and between the same parallels, the area of the triangle is half that of the parallelogram.



GIVEN: $\parallel^m ABDC$ and $\triangle ABE$ on the same base AB and between the same parallels AB and CE.

REQUIRED: To prove $\triangle ABE = \frac{1}{2} \parallel^m ABDC$.

PLAN and CONSTRUCTION: Join AD. Compare $\measuredangle ABD$ and ABE.

PROOF:

STATEMENTS	REASONS
1. $\triangle ABD = \triangle ADC$.	1. Theorem 17, C.
2. $\triangle ABD = \triangle ABE$.	2. Theorem 21.
3. $\triangle ABE = \frac{1}{2} \parallel^m ABDC$.	

Exercises. *Parallelograms and Triangles*

1. Draw any scalene triangle and show how it could be transformed into (1) an isosceles triangle and (2) a right-angled triangle, both of the same area and standing on the same base.

2. Construction 8 may be used to transform a pentagon into a quadrilateral. Write out the construction for this.

3. Two triangles ABC and ABD stand on the same base AB and between the same parallels. AD and BC intersect at O. Prove $\triangle AOC = \triangle BOD$.

4. X is the mid-point of the base BC of $\triangle ABC$. Y is the mid-point of AX. Prove that $\triangle ABY$, BYX, CYX, AYC are equal.

Exercises. *Vocabulary Review*

1. Draw a sketch illustrating each of the following terms:

(1) Circle.
(2) Acute angle.
(3) Parallelogram.
(4) Right-angled triangle.
(5) Rhombus.
(6) Median.
(7) Exterior angle of a triangle.
(8) Diameter.
(9) Hypotenuse.
(10) Obtuse angle.
(11) Equilateral triangle.
(12) Trapezium.
(13) Straight angle.
(14) Pentagon.
(15) Perpendicular bisector.
(16) Diagonal.
(17) Vertex.
(18) Scalene triangle.
(19) Regular pentagon.
(20) Isosceles triangle.
(21) Interior angle of a triangle.

2. Define the following:

(1) Similar triangles that are not congruent.
(2) Vertically opposite angles.
(3) Supplementary angles.
(4) Perpendicular lines.
(5) Congruent triangles.
(6) Complementary angles.
(7) Parallel lines.
(8) Adjacent angles.
(9) Alternate angles.

213. Summary. In this chapter you have learned

Theorems and Constructions:

THEOREM 20. *Parallelograms on the same base and between the same parallels (or of the same altitude) are equal in area.*

COROLLARY 1. *The area of a parallelogram is equal to the product of its base by its altitude.*

COROLLARY 2. *Parallelograms on equal bases and of the same altitude are equal in area.*

THEOREM 21. *Triangles on the same base and between the same parallels are equal in area.*

COROLLARY. *Triangles on equal bases and of the same altitude are equal in area.*

THEOREM 22. *If a triangle and a parallelogram stand on the same base and between the same parallels, the area of the triangle is half that of the parallelogram.*

CONSTRUCTION 8. *To construct a triangle equal in area to a given quadrilateral.*

Test on Chapter XVII

1. Prove that the median of a triangle divides it into two equal triangles.

2. Prove that any line passing through the intersection point of the diagonals of a parallelogram divides the parallelogram into two equal parts.

3. Draw any quadrilateral *PQRS* and construct a triangle equal in area to the quadrilateral. Prove that your construction is correct.

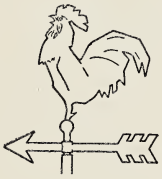
4. Prove that triangles on the same base and between the same parallels are equal in area.

5. Prove that parallelograms on the same base and between the same parallels are equal in area.

CHAPTER XVIII

LOCI

214. Where are All Points in a Plane Located that are a Fixed Distance from a Given Point? There are many examples of points that have a certain freedom of movement but that always remain a fixed distance from some other point. For example, the path traced by any point on the hands of the clock is a circle. The given point is in this instance the centre of the clockface.



1. What path is traced by a point on a weather vane?

2. What path is traced by the tip of the pendulum of the clock?

3. What is the path of an outer corner of a door as it swings open and shut?

4. P and Q are two points 12 inches apart. If P is fixed and Q is free to move in the same plane but is always exactly 12 inches from P, what kind of *track* will it make?

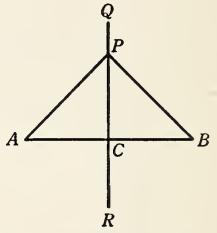
5. Are there any points 12 inches from P which are not on the circumference of the circle?

The word *locus* means "place." It is much like the word *location*. When we say *Find the locus*, we mean *Find the location of* or *Locate the points*.

The locus of points 12 inches from P is a circle with centre at P and radius 12 inches, if we are finding a plane locus.

In sections 215, 216, 217, 218, and 219 the discussion is limited to loci in a plane.

215. What is the Locus of Points Equally Distant from Two Given Points? A and B are the two given points. Join AB. Take C the middle point of AB.



1. Let P be any other point equidistant from A and B. Join PA, PB, PC.

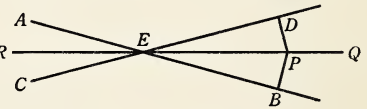
2. It can be shown that $\angle ACP = \angle BCP = 90°$. How?

3. What is true of every point on the perpendicular bisector of AB?

4. Are any points equidistant from A and B not on QR?

5. What is the locus of points that are equidistant from any two given points?

216. What is the Locus of Points that are Equidistant from Two Intersecting Straight Lines? 1. Mark P as the position of any point that is equidistant from the two straight lines AB, CD that intersect in E.



2. Drop perpendiculars from P to AB and CD. Join EP.

3. Prove $\angle PED = \angle PEB$. Use Theorem 14.

4. Produce EP in both directions.

5. Are all points on RQ equidistant from AB and CD?

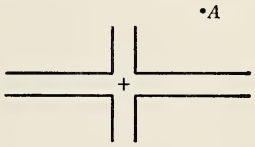
6. Do all points that are equidistant from AB and CD lie on RQ? Check your answer by bisecting $\angle AED$ or $\angle CEB$.

7. What is the locus of points that are equidistant from two intersecting straight lines?

THEOREM 23

Any point in the bisector of an angle is equidistant from the arms of the angle.

217. Locating a Treasure Chest. Locate a treasure chest buried 100 feet from the crossroads and equidistant from the two trees A and B.



1. How many requirements are to be met by this locus?

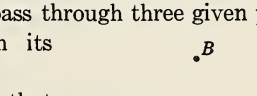
2. Solve each requirement separately.

3. Where might the treasure be to satisfy only the first requirement? (Draw the figure in your notebook.)

4. Where might the treasure be to satisfy only the second requirement?

5. Where must the treasure be to satisfy both requirements? How many positions for the treasure are possible?

218. Locating a Point that is Equidistant from Three Given Points. This problem is the same as that of finding the centre of a circle that will pass through three given points.



1. Copy the figure with its three points A, B, and C.

2. The first requirement is that the required point P is to be equidistant from A and B. What is the second requirement?

3. Is there any other requirement that is not satisfied when these two conditions are satisfied?

4. Where must P be to meet the first requirement?

5. Where must P be to meet the second requirement?

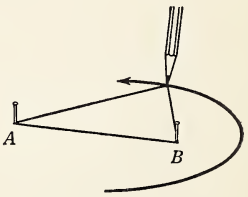
6. What is the locus of a point that is equidistant from three given points?

7. Is it possible to find a point equidistant from three given points if the three points are in a straight line?

219. Definition of Locus. The *locus* (plural *loci*) of a point is the path traced by it when its positions are determined by some fixed law or laws.

Exercises. *Loci*

1. Place two pins upright as at A and B. Loop a string with ends tied around A and B. Hold loop taut and trace with pencil through all possible positions.



(1) What is the path traced by the pencil?

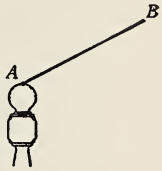
(2) What is the law in this case? Note that the length of AB does not change and that the length of the string does not change.

2. Drawing to scale, mark the position of two trees 60 feet apart. Find a point 20 feet from one tree and 70 feet from the other. How many positions meet both requirements?

3. A package is dropped from an airplane and falls vertically. What is the locus of the centre of the package?

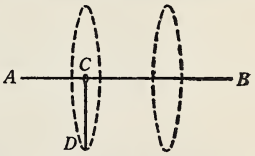
4. What is the locus of points that are 1 inch from the edge of this paper?

220. Loci in Space. So far we have considered loci in one plane only. Let us now consider some other cases.



Example 1. B is a point 12 inches from A, and A is fixed in position. B is free to move anywhere in any plane, provided only that it is always 12 inches from A. What is the locus of B?

Example 2. AB is a taut wire. CD is a wire looped about AB and free to swing about or slide along AB.

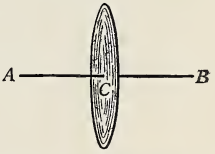


1. What is the locus of D if CD swings about AB but does not slide along it?

2. What is the locus of D if CD hangs vertically, slides along AB, but does not swing?

3. What is the locus of D if free to both swing and slide?

Example 3. In section 215 we considered the locus of a point that is always in one plane and is equally distant from two points. What is the locus of a point that is always equidistant from two given points, provided we do not require the moving point to remain on one fixed plane?



C is a disc free to rotate on the axis AB. A and B are equidistant from C, the centre of the disc.

1. How many points on the disc are equidistant from A and B?

2. Imagine the radius of the disc greatly increased. What is the locus of all points equidistant from A and B?

221. Summary. In this chapter you have learned

(1) The meaning of the word *locus*.

(2) About the following loci on a plane:

(*a*) Locus of points at a fixed distance from one given point.

(*b*) Locus of points at a fixed distance, or fixed distances, from two given points.

(*c*) Locus of points at a fixed distance, or fixed distances, from three given points.

(*d*) Locus of points equidistant from two intersecting straight lines. This is given as

THEOREM 23. *Any point in the bisector of an angle is equidistant from the arms of the angle.*

(3) A few facts regarding loci in three-dimensional space.

Test on Chapter XVIII

1. What is the locus of a point that is a fixed distance from a given plane?

2. What is the locus of points equidistant from two parallel lines?

3. What is the locus of the perimeter of a rectangle which rotates about one of its sides? of the circumference of a circle about its diameter?

4. A straight line is moved in a direction perpendicular to its own length. What is its locus?

5. What is the locus traced by the circumference when a circle moves in a direction perpendicular to its diameter?

6. What is the locus when a square is moved in a direction perpendicular to its own plane?

7. What is the locus of points at a given distance from a straight line and at a given distance from a fixed point not on the line?

CHAPTER XIX

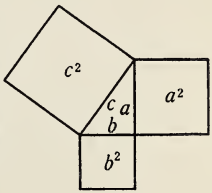
THEOREM OF PYTHAGORAS

222. The Hypotenuse Rule. In any right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides. This statement is often called the *Hypotenuse Rule.* Since it is said to have been proved for any right-angled triangle by the Greek mathematician Pythagoras (pĭ-thăg′ō-răs), who lived about 550 B.C., it is also called the *Pythagorean Relation.*

Pythagoras is said to have been a pupil of Thales and to have visited Egypt. He settled in Crotona, a Greek city of Italy. There he established a famous school in which arithmetic, geometry, music, and astronomy were the principal studies.

From the earliest times the right angle has played an important part in such practical arts as building, surveying, weaving, and the like. It is not surprising, therefore, that interesting facts about right angles, and hence about right-angled triangles, squares, and rectangles, became known many centuries ago.

NOTE. If the length of the hypotenuse of a right triangle is represented by c, and the lengths of the other two sides are represented by a and b, as suggested in the figure at the right, the Hypotenuse Rule may be expressed as follows: $c^2 = a^2 + b^2$. This relation may also be written in the form $a^2 = c^2 - b^2$, or $b^2 = c^2 - a^2$.



The proof of the Hypotenuse Rule is given with the following theorem:

THEOREM 24

In any right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

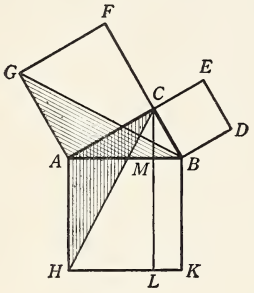


FIG. 1

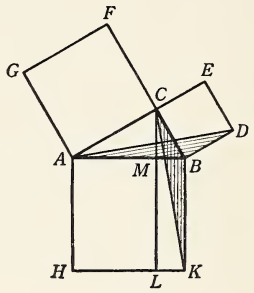


FIG. 2

GIVEN: ABC is a triangle with a right angle at C.

REQUIRED: To prove that the square on AB is equal to the sum of the squares on AC and CB.

PLAN and CONSTRUCTION: Draw perpendicular CL from C to HK. Prove that square on AC is equal to rectangle $AMLH$ and that square on CB is equal to rectangle $BMLK$.

PROOF: (Step 1: That $\triangle ABG = \frac{1}{2}$ square on AC) (Fig. 1)

STATEMENTS	REASONS
1. $\triangle ABG$ and square $ACFG$ are on the same base AG and between the same parallels, GA and FB.	
2. $\triangle ABG = \frac{1}{2}$ square $ACFG$.	2. Theorem 22.

(Step 2: That $\triangle AHC = \frac{1}{2}$ rectangle $AHLM$)

STATEMENTS	REASONS
3. $\triangle AHC$ and rectangle $AHLM$ are on the same base AH and between the same parallels, AH and CL.	
4. $\triangle AHC = \frac{1}{2}$ rectangle $AHLM$.	4. Theorem 22.

(Step 3: That $\triangle ABG = \triangle AHC$)

STATEMENTS	REASONS
5. Compare $\triangle ABG$ and AHC.	
6. $\angle GAB = 90° + \angle CAB$.	
7. $\angle HAC = 90° + \angle CAB$.	
8. $\angle GAB = \angle HAC$.	8. Things equal to the same thing are equal.
9. $GA = AC$.	9. Sides of same square.
10. $AB = AH$.	10. Sides of same square.
11. $\triangle ABG = \triangle AHC$.	11. Theorem 11.

(Step 4: That square on AC = rectangle $AHLM$)

STATEMENTS	REASONS
12. Square on AC $= 2 \triangle ABG$ $= 2 \triangle AHC$ = rectangle $AHLM$.	12. Already proved.

By using Fig. 2 it may be proved similarly that the square on $BC = 2 \triangle ABD = 2 \triangle KBC$ = rectangle $BKLM$.

By addition, square on AC + square on BC = rectangle $AHLM$ + rectangle $BKLM$ = square on AB.

223. First Application of the Hypotenuse Rule. The Hypotenuse Rule is one of the most important principles of mathematics. It can be shown that the right-angled triangle is the only triangle in which the square on the longest

side is exactly equal to the sum of the squares on the other two sides. This fact makes it possible to construct a right-angled triangle and hence a right angle without using a protractor or any other angle-measuring instrument.

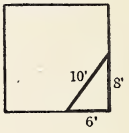
The following exercises will illustrate this important application.

Exercises. *The Hypotenuse Rule*

1. Show that the following sets of values may be used to represent the sides of a right-angled triangle:

(1) 6, 8, 10. (2) 5, 12, 13. (3) 7, 24, 25. (4) 9, 12, 15.

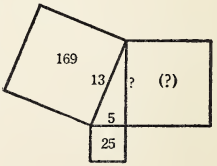
2. Construct a triangle whose base is 5 inches long and whose other two sides are 3 inches and 4 inches long. Measure the angle opposite the longest side. How large should this angle be?



3. In outlining the corners of a rectangular building, builders sometimes use three poles whose lengths are 6 feet, 8 feet, and 10 feet, as suggested by the diagram at the right. Can you explain why this method is correct? Test by a scale drawing.

224. Second Application of the Hypotenuse Rule. The Hypotenuse Rule also makes it possible to find indirectly the value of any side of a right-angled triangle when the values of the other two sides are known.

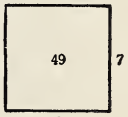
For example, suppose that the hypotenuse of a right-angled triangle is known to be 13 inches, while one leg is 5 inches. How long is the other leg? (*Leg* means either of the two shorter sides of a right-angled triangle.)



Using the Hypotenuse Rule, we have $169 = 25 + (?)$.

That is, the square on the third side is seen to have an area of 144 square inches. (Explain.) Hence its side must be 12 inches, since $12 \times 12 = 144$.

225. Square Root. Before we can readily apply the Hypotenuse Rule in finding unknown distances, we must learn how to find the *side* of a square when we know the area of the square. In many cases this can be done by *inspection*. Thus, if the area of a square is 49, the side of the square is 7, since $7 \times 7 = 49$.



There are many squares, however, whose sides can *not* be found by inspection. The process by which we obtain such values is called *the process of finding a square root.* It corresponds to the process of finding the side of a square whose area is known. That is,

The square root of any given number is that number which when multiplied by itself produces the given number.

To show that the square root of a number is to be found, a special sign is used, which is called the *radical sign* ($\sqrt{\ }$).

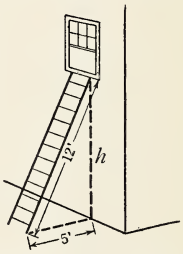
Thus, $\sqrt{4} = 2$ is read "the square root of 4 is 2."
It is evident that $\sqrt{9} = 3$, since $3 \times 3 = 9$, or $3^2 = 9$.
Again, $\sqrt{1600} = 40$, since $40 \times 40 = 1600$, or $40^2 = 1600$.

226. Finding Square Roots from a Table. Since the square roots of numbers have to be used in many practical problems, it saves a good deal of time to obtain such roots directly from a table like the one shown on page 351. This table makes it possible to find quickly the squares of numbers from 1 to 150, and also their square roots to the nearest thousandth. You will see at once how the table is to be used.

Thus, the *square* of 17 is 289, while $\sqrt{17}$ is 4.123. Again, the *square* of 91 is 8281, while $\sqrt{91}$ is 9.539. Also, any number in a column headed by the word "Squares" has its square root in the same row in the column headed "No." Thus, locate 3969 in the second column of "Squares." Its square root is seen to be 63.

We shall now study a simple problem which will test your understanding of the Hypotenuse Rule and of square root.

Illustrative Example. A ladder 12 feet long just reaches to the lower edge of a window when the foot of the ladder is 5 feet from the wall. How far is the window above the ground?



Solution. A right-angled triangle is formed by the ladder, the given distance of 5 feet, and the unknown height. Using the Hypotenuse Rule, we have

$$12^2 = 5^2 + (?)^2,$$

or $$(?)^2 = 12^2 - 5^2;$$

that is, $$(?)^2 = 144 - 25 = 119,$$

or $$(?) = \sqrt{119}.$$

From the table we see that $\sqrt{119} = 10.909$. Hence, to the nearest tenth, the required height is 10.9 feet.

NOTE. If in later exercises you need the square root of a number such as 544, proceed as follows:

$$544 = 2 \times 2 \times 136;\ \text{so}\ \sqrt{544} = 2\sqrt{136} = 2 \times 11.662 = 23.324.$$

Exercises and Problems. *Using the Hypotenuse Rule*

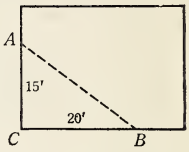
In each of the following exercises let a, b, and c represent the sides of a right-angled triangle, c being used for the hypotenuse. Find the missing side in each case.

1. $a = 6,\ b = 8.$
2. $a = 5,\ b = 12.$
3. $b = 12,\ c = 20.$
4. $a = 8,\ c = 17.$
5. $c = 26,\ a = 10.$
6. $b = 40,\ c = 41.$

TABLE OF SQUARES AND SQUARE ROOTS OF NUMBERS

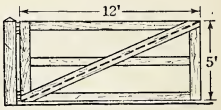
No.	Squares	Square Roots	No.	Squares	Square Roots	No.	Squares	Square Roots
1	1	1.000	51	2,601	7.141	101	10,201	10.050
2	4	1.414	52	2,704	7.211	102	10,404	10.100
3	9	1.732	53	2,809	7.280	103	10,609	10.149
4	16	2.000	54	2,916	7.348	104	10,816	10.198
5	25	2.236	55	3,025	7.416	105	11,025	10.247
6	36	2.449	56	3,136	7.483	106	11,236	10.296
7	49	2.646	57	3,249	7.550	107	11,449	10.344
8	64	2.828	58	3,364	7.616	108	11,664	10.392
9	81	3.000	59	3,481	7.681	109	11,881	10.440
10	100	3.162	60	3,600	7.746	110	12,100	10.488
11	121	3.317	61	3,721	7.810	111	12,321	10.536
12	144	3.464	62	3,844	7.874	112	12,544	10.583
13	169	3.606	63	3,969	7.937	113	12,769	10.630
14	196	3.742	64	4,096	8.000	114	12,996	10.677
15	225	3.873	65	4,225	8.062	115	13,225	10.724
16	256	4.000	66	4,356	8.124	116	13,456	10.770
17	289	4.123	67	4,489	8.185	117	13,689	10.817
18	324	4.243	68	4,624	8.246	118	13,924	10.863
19	361	4.359	69	4,761	8.307	119	14,161	10.909
20	400	4.472	70	4,900	8.367	120	14,400	10.954
21	441	4.583	71	5,041	8.426	121	14,641	11.000
22	484	4.690	72	5,184	8.485	122	14,884	11.045
23	529	4.796	73	5,329	8.544	123	15,129	11.091
24	576	4.899	74	5,476	8.602	124	15,376	11.136
25	625	5.000	75	5,625	8.660	125	15,625	11.180
26	676	5.099	76	5,776	8.718	126	15,876	11.225
27	729	5.196	77	5,929	8.775	127	16,129	11.269
28	784	5.292	78	6,084	8.832	128	16,384	11.314
29	841	5.385	79	6,241	8.888	129	16,641	11.358
30	900	5.477	80	6,400	8.944	130	16,900	11.402
31	961	5.568	81	6,561	9.000	131	17,161	11.446
32	1,024	5.657	82	6,724	9.055	132	17,424	11.489
33	1,089	5.745	83	6,889	9.110	133	17,689	11.533
34	1,156	5.831	84	7,056	9.165	134	17,956	11.576
35	1,225	5.916	85	7,225	9.220	135	18,225	11.619
36	1,296	6.000	86	7,396	9.274	136	18,496	11.662
37	1,369	6.083	87	7,569	9.327	137	18,769	11.705
38	1,444	6.164	88	7,744	9.381	138	19,044	11.747
39	1,521	6.245	89	7,921	9.434	139	19,321	11.790
40	1,600	6.325	90	8,100	9.487	140	19,600	11.832
41	1,681	6.403	91	8,281	9.539	141	19,881	11.874
42	1,764	6.481	92	8,464	9.592	142	20,164	11.916
43	1,849	6.557	93	8,649	9.644	143	20,449	11.958
44	1,936	6.633	94	8,836	9.695	144	20,736	12.000
45	2,025	6.708	95	9,025	9.747	145	21,025	12.042
46	2,116	6.782	96	9,216	9.798	146	21,316	12.083
47	2,209	6.856	97	9,409	9.849	147	21,609	12.124
48	2,304	6.928	98	9,604	9.899	148	21,904	12.166
49	2,401	7.000	99	9,801	9.950	149	22,201	12.207
50	2,500	7.071	100	10,000	10.000	150	22,500	12.247

7. In building the foundation walls for a rectangular house, masons sometimes test the accuracy of the right angle at each corner by a method very similar to that used in ancient Egypt. Thus, in a rectangular ground plan like the one shown in the figure, a mason might measure off $AC = 15$ feet and $BC = 20$ feet. In that case, what would have to be the length of AB if $\angle C$ is to be a right angle?



8. A ladder 25 feet long just reaches the lower edge of a window when the foot of the ladder is 7 feet from the wall. How far is the window above the ground?

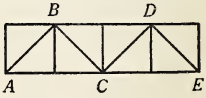
9. A farmer is making a rectangular gate 5 feet high and 12 feet wide. How long should he make the diagonal piece?



10. The rope attached to the top of a flagpole is 65 feet long. When the rope is pulled taut, the loose end just touches the ground 16 feet from the foot of the pole. How high is the flagpole?

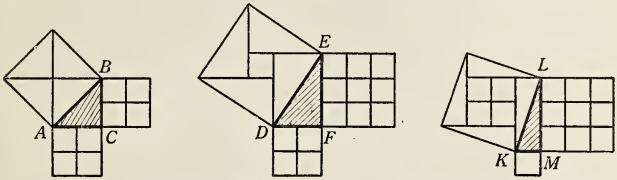
11. What is the longest straight line that can be drawn on a rectangular sheet of paper 8 inches by 15 inches? Draw a rectangle of that size either on paper or on the blackboard and check your result by measurement.

12. If each of the squares in the design at the right represents 1 square inch, how long is the zigzag line from A to E?



13. A tree is broken 20 feet from the ground. The two parts hold together, while the top of the tree touches the ground 12 feet from the foot. Find the original height of the tree.

14. In each of the following figures the small squares represent square inches. In each case find (1) the area of the square on the hypotenuse; (2) the length of the hypotenuse.

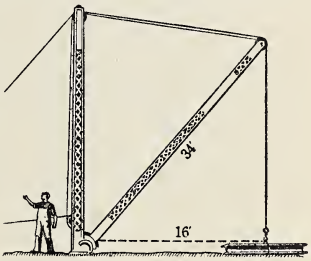


15. Find the diagonal of a rectangle whose dimensions are (1) 8″ and 15″; (2) 11″ and 60″; (3) 13″ and 84″.

16. If a rectangular sheet of paper is 6 inches by 5 inches, how long is a side of a square sheet having the same area?

17. The side of a square baseball diamond is 90 feet. Find, to the nearest tenth of a foot, the distance from second base to the home plate.

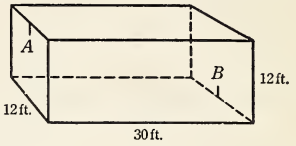
18. A derrick has a movable arm 34 feet long. A weight to be lifted is 16 feet from the foot of the arm. How long is the cable extending from the end of the arm to the weight?



19. Two telephone poles are 60 feet apart. The poles are 42 feet and 30 feet in height respectively. How long is a taut wire connecting their tops?

20. Can an umbrella 28″ long be packed in the bottom of a suitcase 24″ by 16″? Explain.

227. Puzzle: Trace the Spider's Path. A room is 30 feet long, 12 feet wide, and 12 feet high. There is a fly on the middle line of one end wall one foot from the floor (at B). There is a spider on the opposite wall one foot from the ceiling and on the middle line of that wall (at A). The fly remains still and the spider catches it after travelling by the shortest possible route from A to B. How far did the spider travel in going from A to B?



HINT. Make a paper model of the room. Spread the paper out flat on the desk and join the two points A and B by a line lying wholly on the paper. You may have to experiment a little in finding the correct method of spreading out the paper model. It is evident that a light cardboard model of the room might be broken down in several ways. For example, the end walls may be left attached to either side or to the top or bottom. You must so spread out the paper model that the straight line joining A and B lies wholly on the paper.

228. Summary. In this chapter you have learned

(1) The following theorem:

THEOREM 24. *In any right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.*

(2) How to use a table of squares and square roots in applying the formula $c^2 = a^2 + b^2$ and in solving problems based on the Hypotenuse Rule.

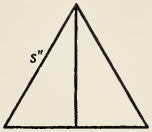
Test on Chapter XIX

1. Interpret the formula $a^2 = c^2 - b^2$ as it is used in connection with a right-angled triangle.

2. Show how the formula $b = \sqrt{c^2 - a^2}$ is obtained.

3. Find the side and the area of a square whose diagonal is 16 inches; 10 inches; 12 inches.

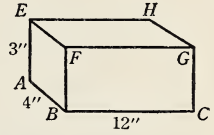
4. If the two sides containing the right angle of a triangle are each one inch in length, the hypotenuse is $\sqrt{2}$ inches in length. Find a method of drawing a line that is $\sqrt{3}$ inches in length.



5. Find the altitude of the equilateral triangle shown at the right.

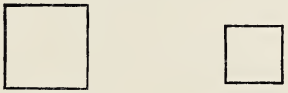
6. Find the area of an equilateral triangle if each side is s inches in length.

7. Find the distance from E to C in the rectangular box shown in the accompanying diagram.



HINT. First find the distance from E to B.

8. Construct a square equal to the sum of two given squares.



9. What is the length of the longest cane that can be placed on the bottom of a trunk 36 inches long and 24 inches wide?

10. What is the length of the longest piece of wire rod that can be placed inside a circular can 12 inches high and having an end radius of $2\frac{1}{2}$ inches?

11. Define and show the pronunciation of each of the following words:

hypotenuse	altitude	loci
locus	diagonal	

12. The base of an isosceles triangle is 16" and each equal side is 13". Find the altitude and area of the triangle.

GENERAL REVIEW

A. ALGEBRA

Complete correctly each of the following statements:

1. The number of miles which an airplane going at the rate of r miles per hour can fly in s hours is _(?)_.

2. The number of pencils that can be bought for b cents, at m cents each, is _(?)_.

3. If x bags of coal weigh y pounds, 2 bags will weigh _(?)_.

4. If $10\,x$ represents the perimeter of a rectangle, and $3\,x$ represents the length, the width is _(?)_.

5. If s pounds of coffee cost t dollars, the number of pounds that can be bought for k dollars is _(?)_.

6. A pupil's standings in four algebra tests were m, n, p, and q. His average standing for these four tests was _(?)_.

7. Find the sum of the terms in each of the following cases:

(1) $9, -8, 15, -7$.
(2) $3\,xy$ and $-8\,xy$.
(3) $-a, -5\,a, 6\,a$.
(4) $100, -50, -70$.
(5) $3\,x^2y, 3\,xy^2, -5\,x^2y$.
(6) $7\,mn^2, -5\,mn^2, 4\,n$.

8. Subtract from $5\,c^2d - 3\,cd + 21$ each of the following expressions:

(1) $9\,c^2d$; (2) $-12\,cd$; (3) 36; (4) -22; (5) $cd + 11$.

Find the product in each case:

9. $3 \times 2 \times (-5)^2$.

10. $(8\,ab) \cdot (-9\,cd)$.

11. $(-4\,a^2b) \cdot (-5\,a^4b^3)$.

12. $(3\,a^2b)(-5\,b)^2(-4\,a^2)$.

13. $6\,a^2b(-9\,a^3b - 5\,ab^2)$.

14. $-8\,n^3(-3\,n - 5\,n^2 + 3\,n^3)$.

Remove parentheses and combine similar terms:

15. $3a + 5(4a - 3b) + 2b - 7(2a + 3b)$.

16. $4w(w + 3) - (7 + 6w) + (4w + 5w^2)$.

17. $-(4a - 7b + 5c) - 3(6b + 2c - 3a)$.

Multiply in each case, and check the result:

18. $(3x^2 + 2x - 4)(3x^2 - 5)$.

19. $(a^2 + 2a - 3)(a^2 - 2a + 3)$.

20. $(S^2 + ST + T^2)(S - T)$.

21. $(x^2 + 2x + 4)(x - 2)$.

Divide in each case, and check the result:

22. $6x^3 - 23x^2 + 29x - 12$ by $2x - 3$.

23. $10x^3 - 9x^2 + 22x - 8$ by $2x^2 - x + 4$.

24. $6x^3 + 11x^2 - 1$ by $3x + 1$.

25. $2x^3 - 7x + 5x^2 + 5$ by $2x - 1$.

If $a = 1$, $b = 2$, $c = 3$, evaluate each of the following expressions:

26. $a^2 + b^2 - c^2 + 6ab - 7ac + 3bc$.

27. $(a + b)^2 + (b - c)^2 + (a - c)^2$.

28. $(a + b - c)^2 - 7(2a + 4b - 3c)^2 + 5abc$.

Solve each of the following equations, and check each solution:

29. $5a - 4 = a$.

30. $5c + 3 = 2c - 9$.

31. $10a = 20 - 5a$.

32. $5x - 3 + 2x = x + 7$.

33. $\frac{5n}{2} = 22 + \frac{7n}{5}$.

34. $\frac{7x}{12} - \frac{1}{4} = 2x - \frac{5}{3}$.

35. $n + 10 + 5n = 37 - 3n$.

36. $8x + 11 - 5x - 2 = 45$.

37. $.05x - 13 = .02x + 2$.

38. $7x - 4 + 5x = 9x - 12 + x$.

39. $\frac{4x}{7} + \frac{11}{5} = \frac{6x}{5}$.

40. $\frac{2x + 4}{4} = \frac{x}{5} + 4$.

41. $11n + 19 = -7n + 10 + 9n$.

Solve the following literal equations for x or y:

42. $7y - 3a^2 = 11a^2$.

43. $5x - 9b = 5c$.

44. $8r - 3(y + 2r) = 5r$.

45. $4(6x - 3a) = 3(5x - a)$.

46. In the formula $i = prt$, if $p = \$500$, $r = .06$, and $t = 2$, find i.

47. By means of the formula $A = \pi r^2$, find the area of a circle whose radius is 5.2.

48. A box is 24 inches long and 18 inches wide. How high must it be to hold 540 pounds of sugar? (One pound of sugar occupies 20 cubic inches.)

49. The milage of telephone wire in Canada, in thousands of miles, for eight recent successive years, was as follows: 3306, 3591, 3983, 4486, 4790, 4985, 5089, 5135. Represent these statistics graphically.

50. If a man swims at the rate of 2 miles an hour, the formula $d = 2t$ represents the relation between the distance (d) and the time (t).

(1) Construct a table that will give the values of d corresponding to $t = 1, 2, 3, 4, 5, 6$.

(2) Plot the graph of this table.

(3) Find from the graph the time needed to cover 7 miles.

51. A family budget allows twice as much for food as for housing, and three times as much for all other items as for housing. If the total budget is \$3600, how much money should be allowed for each item?

52. The sum of two numbers is 12, and their difference is 4. Find the larger number.

53. A man invested \$10,000 in two enterprises. The first pays him 6% on his investment and the second 5%. If his annual income from both investments is \$560, how much did he invest in each?

54. I changed a 5-dollar bill and received in return 35 coins consisting of quarters and dimes. How many of each kind did I receive?

55. Two trains start at the same time from stations that are 288 miles apart and travel toward each other. They meet in 4 hours from the time they started. If the rate of one is 6 miles an hour less than the rate of the other, what is the rate of each train?

56. How many pounds of 45-cent coffee and how many pounds of 25-cent coffee must be mixed together to make 50 pounds worth 33¢ a pound?

B. GEOMETRY

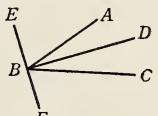
I. Exercises

1. Construct a quadrilateral $ABCD$ with $AB = 1\frac{1}{2}$ inches, $BC = 1\frac{1}{4}$ inches, $CD = 2$ inches, $DA = 3$ inches, and $\angle A = 50°$.

2. Construct a right-angled triangle given the hypotenuse and one side.

3. The four angles at a point have measures that are in the ratio $1 : 2 : 3 : 4$. How large is each angle?

4. GIVEN: BD bisects $\angle ABC$, $EBF \perp BD$.
TO PROVE: $\angle EBA = \angle CBF$.



5. $\measuredangle B$ and C of $\triangle ABC$ are bisected by lines which meet in D. Lines drawn from D parallel to AB and AC cut BC at E and F respectively. Prove BC equal to the perimeter of $\triangle DEF$.

6. Give three statements and their converses which are necessarily true.

7. Give three statements and their converses which are not true.

8. The bisector of an angle of a triangle and the bisector of the adjacent exterior angle are at right angles.

9. An exterior angle of a triangle is equal to the sum of the two interior and opposite angles.

10. P is a point between the arms of an acute angle ABC. Perpendiculars are drawn from P to BA and BC. Prove that the angle between the perpendiculars is the supplement of $\angle ABC$.

11. Complete the following table:

FIGURE	NUMBER OF SIDES	SUM OF ANGLES
Triangle	3	180° = 1 straight angle
Quadrilateral	(?)	(?)
Hexagon	(?)	(?)
Pentagon	(?)	(?)
N-gon	N	(?)

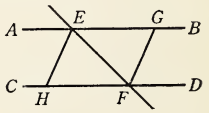
12. Any point equidistant from the arms of an angle is on the bisector of the angle.

13. The perpendiculars from the ends of one side of a triangle to the median drawn to that side are equal.

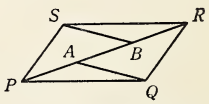
14. The lines AOB, COD bisect each other at O. Prove that $CA = BD$ and $CB = AD$.

15. In a quadrilateral $ABCD$, $\measuredangle A$ and C are equal and $\measuredangle B$ and D are equal. Prove that it is a parallelogram.

16. GIVEN: $AB \parallel CD$,
FG bisects $\angle EFD$,
EH bisects $\angle AEF$.
TO PROVE: $EHFG$ is a parallelogram.



17. GIVEN: $PQRS$ is a parallelogram,
$PB = AR$.
TO PROVE: $SB = AQ$.



18. The difference between two adjacent angles of a parallelogram is 60°. Find each angle of the parallelogram.

19. Prove that (1) the bisectors of the angles of a parallelogram form a rectangle; (2) the bisectors of the angles of a rectangle form a square.

20. $ABCD$ is a quadrilateral. $\angle A = \angle B$, $\angle C = \angle D$. Prove that $AD = BC$.

21. P is a given point between two straight lines AB and AC. Draw a straight line terminated in these lines and bisected at P.

22. Every straight line drawn through the point of intersection of the diagonals of a parallelogram and terminated by the sides of the parallelogram is bisected and bisects the parallelogram.

23. The medians of a triangle trisect each other.

24. $ABCD$ is a parallelogram. $AB \parallel CD$. AD is bisected at E; BC is bisected at F. Prove that BD is trisected when EC and AF are drawn.

25. $ABCD$ is a quadrilateral. Through D and B draw lines parallel to AC. Through A and C draw lines parallel to BD. Prove that the resulting quadrilateral is a parallelogram equal to twice the quadrilateral $ABCD$.

26. $EFGH$ is a parallelogram. HK and FL on HG and FE respectively are cut off equal to each other. Prove that $ELGK$ is a parallelogram.

27. $ABCD$ is a parallelogram with diagonals intersecting at O. P is a point in OC. Prove that $\triangle PAB$ and PAD are equal in area.

28. How many lines of symmetry are there in an equilateral triangle? in a square? in a regular hexagon?

29. Triangles ABC and DEF are similar. $AB = 12$ inches, $AC = 9$ inches, and $BC = 10$ inches. Draw $\triangle DEF$, making DE 4 inches in length.

30. Draw a line 4 inches long and divide it into five equal parts.

31. Construct a quadrilateral $ABCD$, in which $\angle B = 90°$, $AB = 1''$, $BC = 1\frac{1}{4}''$, $CD = DA = 1\frac{1}{2}''$. Then construct a triangle equal in area to $ABCD$.

32. A line parallel to the base BC of a triangle ABC cuts sides AB and AC in D and E. Prove that $\triangle ABE$ and ACD are equal in area.

33. Find the locus of the mid-points of all lines from a given point P to a given line AB.

34. ABC is a triangle. Find a point D in BC such that distances from D to AC and AB are equal.

35. Find a point equidistant from the three vertices of a triangle.

36. Is it possible to find a point which is equidistant from four given points? In what particular cases may such a point be found?

37. What is the locus of points which are at fixed distances from each of two given lines? Solve the problem when the two lines are on a plane. What is the locus if the lines and points may be anywhere in space?

38. A treasure is buried in a field 50 feet from one tree and 30 feet from another tree. The trees are 40 feet apart. Draw a plan showing the possible positions of the treasure. How many feet is the treasure from the straight path connecting the two trees?

39. ABC is a triangle with A a right angle. D is any point on AC. Prove $BD^2 + AC^2 = BC^2 + AD^2$.

40. State the geometrical fact or principle applied in each of the following:

(1) A tennis court is being marked out to be 78 feet long and 36 feet wide. A man measures the diagonal to see whether it is 85.9 feet.

(2) A board is nailed diagonally across a gate to keep it from sagging.

(3) Parallel rulers.

41. Complete each of the following:

(1) If $a = b$ and a is greater than c, then b is $_(?)_$ c.

(2) If $a = b$ and $c = d$, then $a - c$ $_(?)_$ $b - d$.

(3) If $X + 3$ is equal to $_(?)_$, then $X = 5$.

42. Complete the data necessary to show the length of the hypotenuse (inches) of a triangle when the lengths (inches) of the two sides of the triangle are as indicated along the lower and left-hand edges of the diagram.

Length of other side	(1)	(2)	(3)	(4)	(5)	(6)	(7)
4							
3				5			
2							
1	$\sqrt{2}$						
	1	2	3	4	5	6	7

Length of one side of a right-angled triangle

43. Fig. 1 is the end view of a barn. The dimensions are as indicated in the diagram. Find the length of ED.

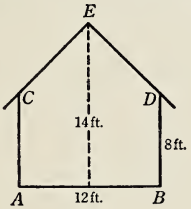


FIG. 1

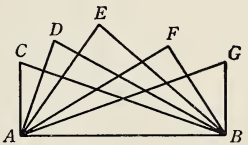


FIG. 2

44. In Fig. 2 the angles CAB, AEB, and GBA are right angles. Which of the following statements are true?

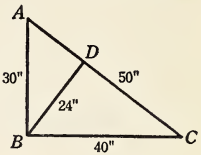
(1) Square on AB = square on AC + square on CB.
(2) Square on AB = square on AD + square on DB.
(3) Square on AB = square on AE + square on EB.
(4) Square on AG = square on AB + square on BG.
(5) Square on CB = square on AB − square on AC.
(6) Square on GB = square on AG − square on AB.

45. Using the formula $c^2 = a^2 + b^2$ and the table of square roots,

(1) Find a when $b = 8$ inches and $c = 10$ inches.
(2) Find b when $a = 9$ inches and $c = 12$ inches.
(3) Find c when $a = 6$ inches and $b = 8$ inches.

46. Find the length of the diagonal of a rectangle whose sides are 16 feet and 12 feet.

47. Find the length of DC in the diagram at the right. $AB = 30$ inches, $BD = 24$ inches, $BC = 40$ inches, and $AC = 50$ inches.



II. Additional Practice on Essentials

Below are three types of exercises that provide varied practice. Teachers may add several examples of each type.

Type 1. Practice in distinguishing between the *If it is true* and the *then it is true* parts of an argument.

With each figure below, certain facts are given as true. Following the given facts are two statements separated by the words "because" and "therefore." You are to decide which of the two words is the correct one. For example, in the statement "I shall help him (because therefore) he has often helped me," we choose the word "because."

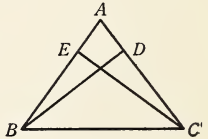
1. GIVEN:

(1) ABC is an isosceles triangle with BC as base.

(2) BD is perpendicular to AC.

(3) CE is perpendicular to AB.

$\angle EBC = \angle DCB$ (because therefore) $AB = AC$.

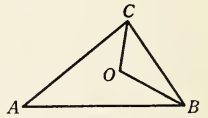


2. GIVEN:

(1) OB bisects $\angle ABC$.

(2) OC bisects $\angle ACB$.

(3) The sum of the three angles of $\triangle ABC$ is 180°.



The sum of $\measuredangle OBC$ and OCB is less than one right angle (because therefore) the sum of $\measuredangle ABC$ and ACB is less than two right angles.

Type 2. Criticizing figures.

1. Criticize Fig. 1 below. *D is a point in the base BC of a triangle, and . . .*

2. Criticize Fig. 2 at the right. *BAC is a triangle having the angle A double the angle B. If . . .*

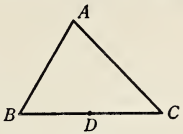


FIG. 1

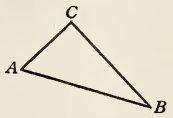
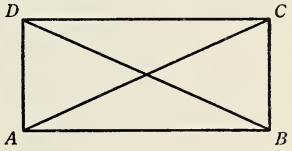


FIG. 2

Type 3. Arranging steps of solution in correct order.

In the following exercise the various steps in the proof are given in the wrong order. Working in your notebook, arrange these steps in a natural order and give the reasons for each statement.

If the diagonals of a parallelogram are equal, the figure is a rectangle.



GIVEN: Parallelogram $ABCD$ with $AC = DB$.

REQUIRED: To prove that $ABCD$ is a rectangle.

PROOF:

(1) Therefore $\parallel^m ABCD$ is a rectangle.
(2) Therefore $\triangle ABD$, ABC are congruent.
(3) $DB = AC$.
(4) Therefore $\angle DAB = \angle ABC$.
(5) $\angle DAB + \angle ABC = 180°$.
(6) Hence $\angle DCB = 90° = \angle ADC$.
(7) Therefore $\angle DAB = \angle ABC = 90°$.
(8) AB is common to each.
(9) Compare $\triangle ABD$, ABC.
(10) $AD = BC$.

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